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MORE ENCOMPASSING ALGORITHM FOR APPROXIMATE SOLUTIONS OF SOME NONLINEAR OPERATOR EQUATIONS

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ABSTRACT. Whenever a closed form solution for a given problem is not readily available, it is of interest to seek for means of obtaining approximate solution through well-defined iterative approach. This work focuses on provision of an iterative method for approximating a common element of set of fixed points of continuous pseudocontractive mapping, set of zeros of inverse strongly monotone mapping, set of solutions of equilibrium problem, and set of common fixed points of countable infinite family of nonexpansive mappings which is a unique solution of a variational inequality problem in the framework of Hilbert space. The iterative method introduced extends, generalizes, improves and unifies some existing results.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|.\|$. A linear map $A : D(A) \subseteq H \to R(A) \subseteq H$ is said to be **γ-strongly positive** if and only if there exists a constant $\gamma > 0$ such that

$$\forall\, x \in D(A), \langle x, Ax \rangle \geq \gamma \|x\|^2.$$

A map $T : D(T) \subseteq H \to R(T) \subseteq H$ is called **nonexpansive** if and only if

$$\forall\, x, y \in D(T),\ \|Tx - Ty\| \leq \|x - y\|.$$

A map $T : D(T) \subseteq H \to R(T) \subseteq H$ is called **a strict contraction** (or simply a contraction) if and only if there exists a constant $\lambda \in [0, 1)$ such that

$$\forall\, x, y \in D(T),\ \|Tx - Ty\| \leq \lambda \|x - y\|.$$

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It is obvious that every contraction is nonexpansive. A map $T : D(T) \subseteq H \to R(T) \subseteq H$ is called **pseudocontractive** if and only if

$$\forall\, x, y \in D(T),\ \langle x-y, Tx-Ty\rangle \le \|x-y\|^2.$$

It can easily be verified that every nonexpansive map is pseudocontractive. A map $B : D(B) \subseteq H \to R(B) \subseteq H$ is called **monotone** if and only if

$$\forall\, x, y \in D(B),\ \langle x-y, Bx-By\rangle \ge 0.$$

The map B is called **η-inverse strongly monotone** if and only if there exists a constant $\eta > 0$ such that

$$\forall\, x, y \in D(B),\ \langle x-y, Bx-By\rangle \ge \eta\|Bx-By\|^2.$$

It is obvious that every η-inverse strongly monotone map is monotone.

Examples abound to show that not every nonexpansive map is a contraction; not every pseudocontractive map is nonexpansive; and not every monotone map is η-strongly monotone. It is worthy to remark that theory of pseudocontractive maps and that of monotone maps are intimately connected in the sense that a map T is pseudocontractive if and one if the map $A := I - T$ is monotone. This can be shown easily. Note that here, I denotes the identity map on H.

Let $T : D(T) \subseteq H \to R(T) \subseteq H$ and $A : D(A) \subseteq H \to R(A) \subseteq H$ be two maps. A point $x^* \in D(T)$ is called a *fixed point* of the map T if and only if $T(x^*) = x^*$; while a point $u^* \in D(A)$ is called a *zero* of the map A if and only if $A(u^*) = 0$. In what follows, the set of fixed points of a map T shall be denoted by $Fix(T)$, while the set of zeros of a map A shall be denoted by $A^{-1}(0)$. Note that $Fix(T) = \{x \in D(T) : Tx = x\}$ and $A^{-1}(0) = \{u \in D(A) : Au = 0\}$.

Let C be a closed convex nonempty subset of a real Hilbert space H and let $A : C \to H$ be a map. A *variational inequality problem* for A is a problem of finding $x^* \in C$ such that for all $y \in C$,

$$\langle Ax^*, y - x^*\rangle \ge 0.$$

The set of solutions of the variational inequality problem for A denoted by $VI(A,C)$ is the set

$$VI(A,C) = \{x \in C : \langle Ax, y-x\rangle \ge 0\ \forall\, y \in C\}.$$

Let $f : C \times C \to \mathbb{R}$ be a bifunction, that is, the map f is such that for any $x \in C$, $f(x,x) = 0$. An *equilibrium problem* (EP) for f (see Blum and

Oettli [8]) is to find $u^* \in C$ such that for all $y \in C$,

$$f(u^*, y) \geq 0. \tag{1.1}$$

The set of solutions of the equilibrium problem for f denoted by $EP(f)$ is the set

$$EP(f) = \{u \in C : f(u,y) \geq 0 \; \forall \, y \in C\}.$$

Due to numerous applications of fixed point techniques, theory of zeros of nonlinear maps, and the fact that many physically significant problems can be expressed as equilibrium problem or variational inequality problem, it has been of great research interest to introduce iterative algorithms for approximation of solutions of fixed point, equilibrium and variational inequality problems; and to find out under what control conditions the sequences generated by the algorithms converge. Several authors had studied various kinds of iterative algorithms and approximation techniques for approximate solutions of nonlinear problems that are intimately connected with equilibrium problems, fixed point problems and variational inequality problems (see, for example, [1]-[7], [17]-[20], [23], [24], [26], [31] and [37]).

Yamada [34] proved the following theorem:

Theorem 1.1. *(Yamada [34]) Let $T : H \to H$ be a nonexpansive map with $Fix(T) \neq \emptyset$. Suppose that a map $A : H \to H$ is L-Lipschitzian and η-strongly monotone with constant $\eta > 0$. Then, for any $\mu \in (0, \frac{2\eta}{L^2})$ and any sequence $\{\lambda_n\}_{n\geq 1} \subset (0,1]$ satisfying*

$$(i) \;\; \lim_{n\to\infty} \lambda_n = 0 \quad (ii) \;\; \sum_{n\geq 1} \lambda_n = +\infty \quad (iii) \;\; \lim_{n\to\infty} (\lambda_n - \lambda_{n+1})\lambda_{n+1}^{-2} = 0,$$

the sequence $\{x_n\}_{n\geq 0}$ generated by

$$x_0 \in H, \;\; x_{n+1} = Tx_n - \mu\lambda_n A(Tx_n), n \geq 0 \tag{1.2}$$

converges strongly to the unique solution $x' \in Fix(T)$ of the variational inequality

$$\langle Ax', y - x' \rangle \geq 0, \; y \in F(T).$$

Since Yamada's hybrid steepest descent method (1.2), several researchers (see for example [9, 16, 20, 21, 26, 30, 33, 40, 41]) have developed iterative methods for solving variational inequality problems.

In 2006, Marino and Xu [21], proved the following theorem:

Theorem 1.2 (Marino and Xu [21])**.** *Let H be a real Hilbert space. Consider a nonexpansive mapping $T : H \to H$ with $Fix(T) \neq \emptyset$, let $g : H \to H$ be a strict contraction with coefficient $\alpha \in (0,1)$ and A be a*

γ^*-strongly positive linear bounded operator. Let $0 < \gamma < \frac{\gamma^*}{\alpha}$. Then, for any sequence $\{\alpha_n\}_{n\geq 0} \subset (0,1)$ satisfying the control conditions:

$$(i)\ \lim_{n\to\infty} \alpha_n = 0,\ \ (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty \ \ and\ \ (iii)\ \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

the sequence $\{x_n\}_{n\geq 1}$ generated iteratively by

$$x_0 \in H,\ \ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n A)Tx_n, n \geq 0 \tag{1.3}$$

converges strongly to the unique solution $x' \in F(T)$ of the variational inequality

$$\left\langle (\gamma g - A)x', y - x' \right\rangle \leq 0,\ y \in F(T).$$

Tian [30], introduced the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_0 \in H,\ \ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \mu \alpha_n A)Tx_n, n \geq 0 \tag{1.4}$$

He proved that if $g : H \to H$ is a contraction with coefficient $\alpha \in (0,1)$, $A : H \to H$ is an η-strongly monotone and $k-$Lipschitzian map, $T : H \to H$ a nonexpansive mapping, μ *and* γ are two constants such that $\mu \in (0, \frac{2\eta}{k^2})$ and $0 < \gamma < \frac{\mu}{\alpha}(\eta - \frac{\mu k^2}{2})$ and the sequence $\{\alpha_n\}_{n\geq 1}$ satisfies appropriate conditions, then the iterative sequence (1.4) converges strongly to a unique solution $x' \in Fix(T)$ of the variational inequality

$$\left\langle (\gamma g - \mu A)x', y - x' \right\rangle \leq 0 \; for\ all\ y \in Fix(T).$$

We observe that the iterative algorithms (1.2), (1.3) and (1.4) converge to an element which is in the fixed point set of a single nonexpansive map T.

Definition 1.3. (See [29]) Let C be a convex nonempty subset of a real Banach space. Let $\{T_j\}_{j\geq 1}$ be a countable infinite family of nonexpansive mappings of C into itself and let $\{\lambda_j\}_{j\geq 1}$ be a sequence of real numbers such that $\lambda_j \in (0,1)$ for all $j \in \mathbb{N}$. For all $n \in \mathbb{N}$, define a mapping $W_n : C \to C$ by

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I, \qquad (1.5)
\end{aligned}$$

The mapping $W_n, n \in \mathbb{N}$ is called the $W-mapping$ generated by the countable infinite family of nonexpansive mappings $T_1, T_2, T_3, \cdots, T_j, \cdots$.

With $W_n, n \in \mathbb{N}$ as in definition 1.3, Yao *et al.* [35] introduced an iterative sequence which converges to a common element of the fixed point sets of an infinite family of nonexpansive mappings in real Hilbert space. They proved the following theorem:

Theorem 1.4. *(Yao et al. [35]) Let H be a real Hilbert space, let $\{T_n\}_{n\geq 1}$ be an infinite family of nonexpansive mappings on H such that the common fixed points set $F = \bigcap\limits_{n\geq 1} Fix(T_n) \neq \emptyset$. Let A be a strongly positive bounded linear operator on H and $f : H \to H$ be a contraction with a contractive constant $\alpha \in [0,1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ satisfying the control conditions:*

$$(i) \lim_{n\to\infty} \alpha_n = 0,\ (ii) \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } (iii)\, 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

then the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_1 \in H,\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) W_n x_n \qquad (1.6)$$

$($where $\gamma > 0$ is some constant$)$ converges strongly to some $x^* \in F$, which is a unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x^* - y \rangle \leq 0,\ y \in F.$$

Colao and Marino [14] established that if $\{x_n\}_{n\geq 1}$ *and* $\{u_n\}_{n\geq 1}$ are sequences generated by $x_1 \in H$,

$$G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, y \in H;$$

$$x_{n+1} = \varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)W_n u_n, n \geq 1, \tag{1.7}$$

where f is a strict contraction with coefficient $\alpha \in (0,1)$, A is a strongly positive bounded linear operator with coefficient $\gamma^* > 0$ and G is an equilibrium function. Let $0 < \gamma < \gamma^* \alpha$ and the sequences $\{\varepsilon_n\}, \{\beta_n\}$ *and* $\{r_n\}$ satisfy appropriate conditions, then both $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ converge strongly to an element $x^* \in \bigcap_{j=1}^{\infty} Fix(T_j) \cap EP(G)$, which is also the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, y - x^* \rangle \geq 0 \; \forall \, y \in \bigcap_{j=1}^{\infty} Fix(T_j) \cap EP(G)$$

The result obtained in [14] extended the corresponding result of [35] in the sense that the iterative sequence (1.7) converges to a common element of the fixed point of countable infinite family of nonexpansive maps and a solution set of an equilibrium problem while the iterative sequence (1.6) converges to a common element of fixed point set of countable infinite family of nonexpansive maps.

Chamnarnpan and Kuman [10] obtained strong convergence result for finding a common element of the set of fixed points for a continuous pseudo-contractive mapping and the solution set of a variational inequality problem governed by continuous monotone mappings. They proved the following theorem:

Theorem 1.5. *(Chamnarnpan and Kumam [10]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping, $A : C \to H$ is a continuous monotone mapping such that $F = Fix(T) \cap VI(C,A) \neq \emptyset$, $f : H \to H$ is a contraction with a contraction constant $\beta \in [0,1)$ and $B : H \to H$ is an inverse strongly monotone mapping. For each $n \in \mathbb{N}$, let T_{r_n} and F_{r_n} be defined for each $x \in H$ by*

$$\begin{aligned} T_{r_n}(x) &= \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1-r_n)z - x \rangle \leq 0, \forall \, y \in C \right\}, \\ F_{r_n}(x) &= \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \; \forall \, y \in C \right\}, \end{aligned}$$

then the sequence $\{x_n\}_{n\geq 1}$ generated iteratively from arbitrary $x_1 \in C$ by

$$\begin{aligned} y_n &= F_{r_n} x_n \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n B)T_{r_n} y_n, \;\; n \geq 1, \end{aligned} \tag{1.8}$$

converges strongly to an element $x^* \in Fix(T) \cap VI(C,A)$ which is the unique solution of the variational inequality

$$\langle (B-\gamma f)x^*, x-x^* \rangle \geq 0 \,\forall\, x \in Fix(T) \cap VI(C,A),$$

where $\alpha_n \in [0,1]$; $r_n \in (0,\infty)$ and $\delta_n \in [0,1]$ are such that

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(C_2) $\lim\limits_{n\to\infty} \delta_n = 0, \quad \sum\limits_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$

(C_3) $\liminf\limits_{n\to\infty} r_n > 0, \quad \sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

The iterative sequence (1.8) generated in [10] converges strongly to a common element of the set of fixed points of a continuous pseudo-contractive map and the solution set of the variational inequality problem. The result obtained in [10] extended that of [21] from the class of nonexpansive mappings to that of continuous pseudo-contractive mappings.

Motivated by the results of the authors mentioned above and others such as [11, 19, 22, 36], an iterative scheme which is more general than the schemes studied by [10, 14, 21, 30, 34, 35] is studied. The theorem obtained extends, generalizes, improves, unifies the corresponding results of these authors.

2. PRELIMINARY

The following will be helpful in the actualization of the main result of this paper:

Recall that a real normed linear space E is said to be uniformly convex if and only if for any $\varepsilon \in (0,2]$, there exists a $\delta_\varepsilon \in (0,1]$ such that for all $x,y \in E$ with $\|x\| = 1 = \|y\|$, $\|x-y\| \geq \varepsilon$, we have that $\left\|\frac{1}{2}(x+y)\right\| < 1-\delta_\varepsilon$. It is well known that every real Hilbert space is uniformly convex (see, for example, [12] and [39]).

Let C be a closed convex nonempty subset of a uniformly convex real Banach space E. Given a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in C, define (for each $m \in \mathbb{N}$)

$$r_m(y) = \sup\{\|x_k - y\| : k \geq m\}, y \in E,$$

then it is shown in [15] that there exists unique $c_m \in C$ such that

$$r_m(c_m) = \inf\{r_m(y) : y \in C\} = r_m \ \forall\, m \in \mathbb{N},$$

$r_{m+1} \leq r_m$ *and* $0 \leq r_m \ \forall\, m \in \mathbb{N}$, so that $\lim\limits_{m\to\infty} r_m = \inf\limits_{m\in\mathbb{N}} r_m$ exists. Edelstein [15] showed that if $\lim\limits_{m\to\infty} r_m = \inf\limits_{m\in\mathbb{N}} r_m = 0$, then the sequence $\{x_n\}$ converges.

If the sequence $\{c_m\}_{m\geq 1}$ converges, then $c_0 = \lim\limits_{m\to\infty} c_m$ is called **the asymptotic center** of $\{x_n\}_{n\geq 1}$ (with respect to C).

Proposition 2.1 (Sahani and Bose [25])**.** *A point $c_0 \in C$ is asymptotic center of $\{x_n\}_{n\geq 1}$ $($with respect to $C)$ if and only if*

$$\limsup_{n\to\infty} \|x_n - c_0\| = \inf_{z\in C} \limsup_{n\to\infty} \|x_n - z\|\,.$$

Lemma 2.2 (Edelstein [15])**.** *Let C be a closed convex nonempty subset of a uniformly convex real Banach space E, Let $\{x_n\}_{n\geq 1}$ be a bounded sequence in C and $\{c_m\}_{m\geq 1}$ be as in Remark 2, then $c = \lim\limits_{m\to\infty} c_m$ exists. In other words, the asymptotic center c of the sequence $\{x_n\}_{n\geq 1}$ $($with respect to $C)$ exists and is unique.*

Lemma 2.3 (Shimoji and Takahashi [27])**.** *Let C be a closed convex nonempty subset of a strictly convex real Banach space. Let $\{T_j\}_{j\geq 1}$ be a countable infinite family of nonexpansive mappings of C into itself and let $\{\lambda_j\}_{j\geq 1}$ be a real sequence such that $0 < \lambda_j \leq b < 1$ for all $j \in \mathbb{N}$, for some constants $b \in (0,1)$. Then, for all $x \in C$ and $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} U_{n,k}(x)$ exists, where $U_{n,k}$ is as in definition 1.3.*

For $k = 1$ in Lemma 2.3, we define a mapping $W : C \to C$ by

$$W(x) = \lim_{n\to\infty} U_{n,1}(x) = \lim_{n\to\infty} W_n(x),$$

then, by Lemma 2.3, the map W is well-defined; moreover, W has the following property:

Lemma 2.4 (Shimoji and Takahashi [27])**.** *Let C be a closed convex nonempty subset of a strictly convex real Banach space. Let $\{T_j\}_{j\geq 1}$ be a countable infinite family of nonexpansive mappings of C into itself such that $\bigcap\limits_{j=1}^{\infty} F(T_j) \neq \emptyset$ and $\{\lambda_j\}_{j\geq 1}$ be a real sequence such that $0 < \lambda_j \leq b < 1$ for all $j \in \mathbb{N}$ for some constant $b \in (0,1)$. Let W_n and W be as in Remark 2 respectively, then $F(W) = \bigcap\limits_{j=1}^{\infty} F(T_j) = \bigcap\limits_{n=1}^{\infty} F(W_n)$.*

Lemma 2.5 (Colao and Marino [14]). *Let C be a closed convex nonempty subset of a strictly convex real Banach space. Let $\{T_j\}_{j\geq 1}$ be a countable infinite family of nonexpansive mappings of C into itself such that $\bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$ and $\{\lambda_j\}_{j\geq 1}$ be a real sequence such that $0 < \lambda_j \leq b < 1$ $forall$ $j \in \mathbb{N}$ for some constant $b \in (0,1)$. Let $W_n : C \to C$ be as in Remark 2, then*

$$\|W_{n+1}(x) - W_n(x)\| \leq 2\prod_{j=1}^{n} \lambda_j \|x-p\| \ \forall\, x \in C, p \in \bigcap_{j=1}^{\infty} F(T_j).$$

Lemma 2.6. *Let H be a real Hilbert space, then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \ \forall\, x,y \in H.$$

Lemma 2.7 (Xu [32]). *Let $\{\lambda_n\}_{n\geq 0}$ be a sequence of nonnegative real numbers satisfying the following conditions:*

$$\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \sigma_n, n \geq 0;$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$. Suppose that $\sigma_n = \circ(\alpha_n), n \geq 0$ $\big($i.e. $\lim_{n\to\infty} \frac{\sigma_n}{\alpha_n} = 0\big)$ or $\sum_{n=0}^{\infty} |\sigma_n| < +\infty$ or $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$, then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.8 (Suzuki [28]). *Let $\{t_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be two bounded sequences in a real Banach space such that $y_{n+1} = \beta_n y_n + (1-\beta_n)t_n$, $for\ all\ n \geq 0$, where $\{\beta_n\}_{n\geq 0}$ is a real sequence satisfying the condition $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$\limsup_{n\to\infty}(\|t_{n+1} - t_n\| - \|y_{n+1} - y_n\|) \leq 0,$$

then, $\lim_{n\to\infty} \|t_n - y_n\| = 0$.

Lemma 2.9 (Blum and Oettli [8]). *Let C be a closed convex nonempty subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a function satisfying the following conditions:*

(A1) $f(x,x) = 0$ $for\ all$ $x \in C$ (that is, f is a bifunction);
(A2) f is monotone, in the sense that $f(x,y) + f(y,x) \leq 0\ \forall\, x,y \in C$;
(A3) $\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)\ \forall\, x,y,z \in C, t \in [0,1]$;
(A4) the function $y \to f(x,y)$ is convex and lower semicontinuous for all $x \in C$,

then for all $r > 0$ and $x \in H$ there exists $u \in C$ such that

$$f(u,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0 \; \forall \, y \in C.$$

Moreover, if we define a mapping $G_r : H \to 2^C$ by

$$G_r(x) = \left\{ u \in C : f(u,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0 \; \forall \, y \in C \right\}, \text{for all } x \in H,$$

then the following hold:

(1) *G_r is single valued for all $r > 0$*
(2) *G_r is firmly nonexpansive, that is, for all $x, z \in H$* $\|G_r x - G_r z\|^2 \leq \langle G_r x - G_r z, x - z\rangle$
(3) *$Fix(G_r) = EP(f)$ for all $r > 0$*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.10 (Zegeye [38]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T' : C \to H$ be a continuous pseudocontractive mapping, then for all $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, T'z\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leq 0 \; \forall \, y \in C.$$

Lemma 2.11 (Zegeye [38]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T' : C \to C$ be a continuous pseudocontractive mapping, then for all $r > 0$ define a mapping $F_r : H \to C$ by*

$$F_r(x) = \left\{ z \in C : \langle y - z, T'z\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leq 0 \; \forall \, y \in C \right\}, \; x \in H,$$

then the following hold:

(1) *F_r is single valued*
(2) *F_r is firmly nonexpansive type mapping, that is $\forall \, x, y \in H$,* $\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle$
(3) *$Fix(F_r)$ is closed and convex and $Fix(F_r) = Fix(T') \; \forall \, x > 0$.*

The following Lemmas whose proofs can easily be obtained shall also be needed in what follows

Lemma 2.12. *Let C be a closed convex nonempty subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $(A1)$ to $(A4)$; $\psi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $\theta : C \to H$ be a monotone mapping. Let $r > 0$ and let G_r be the mapping in Lemma 2.9, then for all $p, q > 0$ and for all $x \in H$, we have*

$$\left\|G_p x - G_q x\right\| \leq \frac{|p-q|}{p}\Big(\left\|G_p x\right\| + \|x\| \Big).$$

Lemma 2.13. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T' : C \to C$ be a continuous pseudocontractive mapping. For $r > 0$, let $F_r : H \to C$ be the mapping in Lemma 2.11, then for any $x \in H$ and for any $p, q > 0$*

$$\left\| F_p x - F_q x \right\| \leq \frac{|p-q|}{p} \Big(\left\| F_p x \right\| + \|x\| \Big).$$

3. MAIN RESULT

In the sequel, the following assumptions are used:

Assumptions: H is a real Hilbert space; $T' : H \to H$ is a continuous pseudocontractive map; $T_j : H \to H,\ j = 1,2,3\cdots$ is a countable infinite family of nonexpansive maps; $f : H \times H \to \mathbb{R}$ is a bifunction satisfying conditions (A1)-(A4); $g : H \to H$ is a contraction map with constant $k \in [0,1)$; $A : H \to H$ is a strongly positive bounded linear operator with coefficient γ; $B : H \to H$ is an η-inverse strongly monotone mapping; the sequences $\{r_n\}_{n\geq 1}, \{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ *and* $\{\lambda_n\}_{n\geq 1}$ are real sequences such that $r_n > 0$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} r_n = r_0 > 0; 0 < \alpha_n < 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty, 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1; 0 < \lambda_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some constant $b \in (0,1)$. ξ is a real constant such that $0 < \xi < 2\eta$, where $\eta > 0$. $\Omega = Fix(T') \cap EP(f) \cap B^{-1}(0) \cap \bigcap_{j=1}^{\infty} Fix(T_j) \neq \emptyset$. For $r > 0, G_r$ and F_r are as in Lemma 2.9 and 2.11, respectively.

Strong convergence of the sequence $\{x_n\}_{n\geq 1}$ generated iteratively from arbitrary $x_1 \in H$ by

$$x_{n+1} = \alpha_n \gamma g(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) W_n (I - \xi B) F_{r_n} G_{r_n} x_n,\ n \geq 1 \tag{3.1}$$

to a unique solution $x' \in \Omega$ of the variational inequality problem

$$\langle \gamma g(x') - Ax', y - x' \rangle \leq 0 \ \ \forall\, y \in \Omega \tag{3.2}$$

is proved. The following is the main theorem of this paper:

Theorem 3.1. *Let $\{x_n\}_{n\geq 1}$ be given by* (3.1). *Suppose that Assumptions above hold, then $\{x_n\}_{n\geq 1}$ converges strongly to a unique solution $x' \in \Omega$ of the variational inequality problem* (3.2).

Proof. This is broken into several steps:
STEP 1: We show that the sequence $\{x_n\}_{n\geq 1}$ given by (3.1) is bounded.

Observe that for all $x,y \in H$,

$$\begin{aligned} \|(x-\xi Bx)-(y-\xi By)\|^2 &= \|(x-y)-\xi(Bx-By)\|^2 \\ &= \|x-y\|^2 - 2\xi\langle Bx-By, x-y\rangle + \xi^2\|Bx-By\|^2 \\ &\leq \|x-y\|^2 - 2\xi\eta\|Bx-By\|^2 + \xi^2\|Bx-By\|^2 \\ &= \|x-y\|^2 - \xi(2\eta-\xi)\|Bx-By\|^2. \end{aligned} \tag{3.3}$$

Since $0<\xi<2\eta$ we get

$$\|(x-\xi Bx)-(y-\xi By)\| \leq \|x-y\| \ \forall\, x,y \in H,$$

Thus, for $p \in \Omega$,

$$\begin{aligned} \|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 &= \|(I-\xi B)F_{r_n}G_{r_n}x_n - (I-\xi B)p\|^2 \\ &\leq \|F_{r_n}G_{r_n}x_n - p\|^2 \leq \|x_n - p\|^2. \end{aligned}$$

Since $\lim\limits_{n\to\infty}\alpha_n = 0$, let us (without loss of generality) assume that for all $n \in \mathbb{N}$, $\alpha_n < \|A\|^{-1}(1-\beta_n)$. So, for each $x \in H$ such that $\|x\| = 1$, we have

$$\begin{aligned} \langle((1-\beta_n)I-\alpha_nA)x,x\rangle &= (1-\beta_n)-\alpha_n\langle Ax,x\rangle \\ &\geq (1-\beta_n)-\alpha_n\|A\| \\ &> (1-\beta_n)-\|A\|^{-1}(1-\beta_n)\|A\| = 0. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, the operator $(1-\beta_n)I-\alpha_nA$ is a positive bounded linear operator, so that

$$\begin{aligned} \|(1-\beta_n)I-\alpha_nA\| &= \sup_{\|x\|=1}\langle((1-\beta_n)I-\alpha_nA)x,x\rangle \\ &= \sup_{\|x\|=1}(1-\beta_n-\alpha_n\langle Ax,x\rangle) \\ &\leq 1-\beta_n-\alpha_n\gamma \end{aligned}$$

Next, we show by mathematical induction that

$$\|x_n - p\| \leq M = \max\left\{\|x_1-p\|, \frac{1}{\gamma(1-k)}\|\gamma g(p)-Ap\|\right\} \tag{3.4}$$

Observe that for $n=1$, the inequality (3.4) clearly holds. Suppose that the inequality is true for $n=c\geq 1$, we show that it holds for $n=c+1$.

To see this, observe that for $p \in \Omega$ and using (3.1) we obtain that

$$\begin{aligned}
\|x_{c+1}-p\| &= \|\alpha_c \gamma g(x_c)+\beta_c x_c+((1-\beta_c)I-\alpha_c A)\times W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p\| \\
&= \|((1-\beta_c)I-\alpha_c A)(W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p)+\alpha_c(\gamma g(x_c)-Ap) \\
&\quad +\beta_c(x_c-p)\| \\
&\leq (1-\beta_c-\alpha_c\gamma)\|W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p\|+\beta_c\|x_c-p\| \\
&\quad +\alpha_c\|\gamma g(x_c)-Ap\| \\
&= (1-\beta_c-\alpha_c\gamma)\|W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p\|+\beta_c\|x_c-p\| \\
&\quad +\alpha_c\|\gamma g(x_c)-\gamma g(p)+\gamma g(p)-Ap\| \\
&\leq (1-\beta_c-\alpha_c\gamma)\|W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p\|+\beta_c\|x_c-p\| \\
&\quad +\alpha_c\gamma\|g(x_c)-g(p)\|+\alpha_c\|\gamma g(p)-Ap\| \\
&\leq (1-\beta_c-\alpha_c\gamma)\|W_c(I-\xi B)F_{r_c}G_{r_c}x_c-p\|+\beta_c\|x_c-p\| \\
&\quad +\alpha_c\gamma k\|x_c-p\|+\alpha_c\|\gamma g(p)-Ap\| \\
&\leq (1-\alpha_c\gamma+\alpha_c\gamma k)\,\|x_c-p\|+\alpha_c\,\|\gamma g(p)-Ap\| \\
&= (1-\alpha_c\gamma(1-k))\|x_c-p\|+\alpha_c\,\|\gamma g(p)-Ap\| \\
&= (1-\alpha_c\gamma(1-k))\|x_c-p\|+\alpha_c\gamma(1-k)\cdot\frac{1}{\gamma(1-k)}\,\|\gamma g(p)-Ap\| \\
&\leq (1-\alpha_c\gamma(1-k))M+\alpha_c\gamma(1-k)M=M.
\end{aligned}$$

So, (3.4) holds for all $n\in\mathbb{N}$. Hence, $\{x_n\}_{n\geq 1}$ is bounded. So, the following $\{g(x_n)\}_{n\geq1}, \{W_n(I-\xi B)F_{r_n}G_{r_n}x_n\}_{n\geq1}, \{(I-\xi B)F_{r_n}G_{r_n}x_n\}_{n\geq1}, \{F_{r_n}G_{r_n}x_n\}_{n\geq1}$ and $\{G_{r_n}x_n\}_{n\geq1}$ are bounded.

STEP 2: We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Now set $t_n=\frac{1}{1-\beta_n}(x_{n+1}-\beta_n x_n)$, then $x_{n+1}=\beta_n x_n+(1-\beta_n)t_n$. Thus,

$$\begin{aligned}
t_{n+1}-t_n &= \frac{1}{1-\beta_{n+1}}(x_{n+2}-\beta_{n+1}x_{n+1})-\frac{1}{1-\beta_n}(x_{n+1}-\beta_n x_n) \\
&= \frac{1}{1-\beta_{n+1}}(\alpha_{n+1}\gamma g(x_{n+1}) \\
&\quad +\big((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}(I-\xi B)F_{r_{n+1}}G_{r_{n+1}}x_{n+1}\big) \\
&\quad -\frac{1}{1-\beta_n}(\alpha_n\gamma g(x_n)+((1-\beta_n)I-\alpha_n A)W_n(I-\xi B)F_{r_n}G_{r_n}x_n) \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\gamma g(x_{n+1})-AW_{n+1}(I-\xi B)F_{r_{n+1}}G_{r_{n+1}}x_{n+1}\big) \\
&\quad -\frac{\alpha_n}{1-\beta_n}(\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n) \\
&\quad +W_{n+1}(I-\xi B)F_{r_n+1}G_{r_n+1}x_{n+1}-W_n(I-\xi B)F_{r_n}G_{r_n}x_n \\
&\quad +W_{n+1}(I-\xi B)F_{r_n}G_{r_n}x_n-W_{n+1}(I-\xi B)F_{r_n}G_{r_n}x_n.
\end{aligned}$$

Thus, we obtain (for some constant $M_0 > 0$) the following:

$$
\begin{aligned}
\|t_{n+1}-t_n\| - \|x_{n+1}-x_n\| \leq\ & \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right] M_0 \\
& + \|W_{n+1}(I-\xi B)F_{r_n}G_{r_n}x_n - W_n(I-\xi B)F_{r_n}G_{r_n}x_n\| \\
& + \|W_{n+1}(I-\xi B)F_{r_{n+1}}G_{r_{n+1}}x_{n+1} - W_{n+1}(I-\xi B)F_{r_n}G_{r_n}x_n\| \\
& - \|x_{n+1}-x_n\| \\
\leq\ & \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right] M_0 \\
& + 2\prod_{j=1}^{n}\lambda_j \|(I-\xi B)F_{r_n}G_{r_n}x_n - p\| + \|F_{r_{n+1}}G_{r_{n+1}}x_{n+1} \\
& - F_{r_{n+1}}G_{r_n}x_n\| + \left\|F_{r_{n+1}}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\right\| \\
& - \|x_{n+1}-x_n\| \\
\leq\ & \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right] M_0 + 2\prod_{j=1}^{n}\lambda_j \|F_{r_n}G_{r_n}x_n - p\| \\
& + \left\|G_{r_{n+1}}x_{n+1} - G_{r_n}x_n\right\| + \left\|F_{r_{n+1}}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\right\| \\
& - \|x_{n+1}-x_n\| \\
\leq\ & \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right] M_0 + 2\prod_{j=1}^{n}\lambda_j \|F_{r_n}G_{r_n}x_n - p\| \\
& + \left\|G_{r_{n+1}}x_{n+1} - G_{r_{n+1}}x_n\right\| + \left\|G_{r_{n+1}}x_n - G_{r_n}x_n\right\| \\
& + \left\|F_{r_{n+1}}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\right\| - \|x_{n+1}-x_n\| \\
\leq\ & \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right] M_0 + 2\prod_{j=1}^{n}\lambda_j \|F_{r_n}G_{r_n}x_n - p\| \\
& + \frac{|r_{n+1}-r_n|}{r_n} \times [(\|F_{r_n}G_{r_n}x_n\| + 2\|G_{r_n}x_n\| + \|x_n\|)].
\end{aligned}
$$

Since $0 < \lambda_n < b < 1$ for all $n \in \mathbb{N}$ and for some constant $b \in (0,1)$, we have $\lim_{n\to\infty} \prod_{j=1}^{n} \lambda_j = 0$. So, we obtain that $\limsup_{n\to\infty}(\|t_{n+1}-t_n\| - \|x_{n+1}-x_n\|) \leq 0$. Therefore, $\lim_{n\to\infty} \|t_n - x_n\| = 0$ by Lemma 2.8; and we know that $x_{n+1} - x_n = (1-\beta_n)(t_n - x_n)$. Hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

STEP 3: In this step, the following equalities are established:

$$\begin{aligned}\lim_{n\to\infty}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n-x_n\| &= \lim_{n\to\infty}\|G_{r_n}x_n-F_{r_n}G_{r_n}x_n\|\\ &= \lim_{n\to\infty}\|BF_{r_n}G_{r_n}x_n\|\\ &= \lim_{n\to\infty}\|x_n-G_{r_n}x_n\|\\ &= \lim_{n\to\infty}\|x_n-F_{r_n}G_{r_n}x_n\|\end{aligned}$$

$$= \lim_{n\to\infty}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n-F_{r_n}G_{r_n}x_n\| = 0.$$

To see this, we have that for some constant $M_1 > 0$,

$$\begin{aligned}\|x_n-W_n(I-\xi B)F_{r_n}G_{r_n}x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-W_n(I-\xi B)F_{r_n}G_{r_n}x_n\|\\ &= \|x_n-x_{n+1}\|\\ &\quad +\|\alpha_n\gamma g(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n A]\\ &\quad \times W_n(I-\xi B)F_{r_n}G_{r_n}x_n-W_n(I-\xi B)F_{r_n}G_{r_n}x_n\|\\ &\le \|x_{n+1}-x_n\|\\ &\quad +\alpha_n\|\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n\|\\ &\quad +\beta_n\|x_n-W_n(I-\xi B)F_{r_n}G_{r_n}x_n\|\\ &\le \|x_{n+1}-x_n\|+\alpha_n M_1\\ &\quad +\beta_n\|x_n-W_n(I-\xi B)F_{r_n}G_{r_n}x_n\|\end{aligned}$$

Thus,

$$\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n-x_n\| \le \frac{1}{1-\beta_n}(\|x_n-x_{n+1}\|+\alpha_n M_1) \qquad (3.6)$$

Using (3.6) and the fact that $\lim_{n\to\infty}\alpha_n = 0$, we obtain that

$$\lim_{n\to\infty}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n-x_n\| = 0. \qquad (3.7)$$

Using (2) of Lemma 2.9 and for fixed $p\in\Omega$ we have

$$\begin{aligned}\|G_{r_n}x_n-p\|^2 &\le \langle G_{r_n}x_n-p, x_n-p\rangle\\ &= \frac{1}{2}\Big[\|G_{r_n}x_n-p\|^2+\|x_n-p\|^2-\|x_n-G_{r_n}x_n\|^2\Big].\end{aligned}$$

So,

$$\|G_{r_n}x_n-p\|^2 \le \|x_n-p\|^2-\|x_n-G_{r_n}x_n\|^2. \qquad (3.8)$$

Also, using (2) of Lemma 2.11, we have

$$\begin{aligned}\|F_{r_n}G_{r_n}x_n-p\|^2 &\le \langle F_{r_n}G_{r_n}x_n-p, G_{r_n}x_n-p\rangle\\ &= \frac{1}{2}\Big[\|F_{r_n}G_{r_n}x_n-p\|^2+\|G_{r_n}x_n-p\|^2-\|G_{r_n}x_n-F_{r_n}G_{r_n}x_n\|^2\Big].\end{aligned}$$

So,

$$\|F_{r_n}G_{r_n}x_n - p\|^2 \leq \|G_{r_n}x_n - p\|^2 - \|G_{r_n}x_n - F_{r_n}G_{r_n}x_n\|^2 . \qquad (3.9)$$

Using the recursion formula (3.1), Lemma 2.6 and convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma g(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n-p\|^2 \\
&\quad +2\alpha_n\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle \\
&\leq \beta_n\|x_{n+1}-p\|^2+2\beta_n\|x_{n+1}-p\|\|x_n-x_{n+1}\|+\beta_n\|x_{n+1}-x_n\|^2 \\
&\quad +(I-\beta_n)\|(I-\xi B)F_{r_n}G_{r_n}x_n-p\|^2 \\
&\quad +2\alpha_n\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle.
\end{aligned}$$

Thus, using (3.3)

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|(I-\xi B)F_{r_n}G_{r_n}x_n-p\|^2 \\
&\quad +\frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\|+\|x_{n+1}-x_n\|)\|x_{n+1}-x_n\| \\
&\quad +\frac{2\alpha_n}{1-\beta_n}\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle \\
&\leq \|F_{r_n}G_{r_n}x_n-p\|^2-\xi(2\eta-\xi)\|BF_{r_n}G_{r_n}x_n\|^2 \\
&\quad +\frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\|+\|x_{n+1}-x_n\|)\times\|x_{n+1}-x_n\| \\
&\quad +\frac{2\alpha_n}{1-\beta_n}\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle \\
&\leq \|x_n-p\|^2-\|G_{r_n}x_n-F_{r_n}G_{r_n}x_n\|^2-\xi(2\eta-\xi)\|BF_{r_n}G_{r_n}x_n\|^2 \\
&\quad +\frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\|+\|x_{n+1}-x_n\|)\|x_{n+1}-x_n\| \\
&\quad +\frac{2\alpha_n}{1-\beta_n}\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle . \qquad (3.10)
\end{aligned}$$

So, for some real constant $M_2 > 0$ we have

$$\begin{aligned}
\|G_{r_n}x_n-F_{r_n}G_{r_n}x_n\|^2 &+ \xi(2\eta-\xi)\|BF_{r_n}G_{r_n}x_n\|^2 \\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\|+\|x_{n+1}-x_n\|)\|x_{n+1}-x_n\| \\
&\quad +\frac{2\alpha_n}{1-\beta_n}\langle\gamma g(x_n)-AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle \\
&\leq (\alpha_n+\|x_{n+1}-x_n\|)M_2.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}(\|G_{r_n}x_n - F_{r_n}G_{r_n}x_n\|^2 + \xi(2\eta-\xi)\|BF_{r_n}G_{r_n}x_n\|^2) = 0.$$

since $\xi(2\eta-\xi) > 0$, applying sandwich theorem we obtain

$$\lim_{n\to\infty}\|G_{r_n}x_n - F_{r_n}G_{r_n}x_n\| = 0 \tag{3.11}$$

and

$$\lim_{n\to\infty}\|BF_{r_n}G_{r_n}x_n\| = 0.$$

From the second line of inequality (3.10), using (3.8) and the fact that F_{r_n} is firmly nonexpansive (thus nonexpansive), we obtain that for some constant $M_3 > 0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|F_{r_n}G_{r_n}x_n - p\|^2 - \xi(2\eta-\xi)\|BF_{r_n}G_{r_n}x_n\|^2 \\
&\quad + \frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\| + \|x_{n+1}-x_n\|) \times \|x_{n+1}-x_n\| \\
&\quad + \frac{2\alpha_n}{1-\beta_n} \times \langle \gamma g(x_n) - AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle \\
&\leq \|G_{r_n}x_n - p\|^2 + [\|x_{n+1}-x_n\| + \alpha_n]M_3 \\
&\leq \|x_n - p\|^2 - \|x_n - G_{r_n}x_n\|^2 + [\|x_{n+1}-x_n\| + \alpha_n]M_3
\end{aligned}$$

So, for some constant $M_4 > 0$,

$$\begin{aligned}
\|x_n - G_{r_n}x_n\| &\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + [\|x_{n+1}-x_n\| + \alpha_n]M_3 \\
&\leq \|x_{n+1}-x_n\| M_4 + [\|x_{n+1}-x_n\| + \alpha_n]M_3.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|x_n - G_{r_n}x_n\| = 0. \tag{3.12}$$

Observe that

$$\begin{aligned}
\|x_n - F_{r_n}G_{r_n}x_n\| &= \|x_n - G_{r_n}x_n + G_{r_n}x_n - F_{r_n}G_{r_n}x_n\|. \\
&\leq \|x_n - G_{r_n}x_n\| + \|G_{r_n}x_n - F_{r_n}G_{r_n}x_n\|
\end{aligned}$$

Using (3.11) and (3.12) we have

$$\lim_{n\to\infty}\|x_n - F_{r_n}G_{r_n}x_n\| = 0. \tag{3.13}$$

Furthermore, observe that

$$\begin{aligned}
\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\| &\leq \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\| \\
&\quad + \|x_n - F_{r_n}G_{r_n}x_n\|.
\end{aligned}$$

Using (3.7) and (3.13), we have

$$\lim_{n\to\infty}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\| = 0.$$

STEP 4: We show that

$$\lim_{n\to\infty} \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| = 0.$$

To see this, let $p \in \Omega$ be fixed. Then,

$$\begin{aligned}
\|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 &= \|F_{r_n}G_{r_n}x_n - \xi BF_{r_n}G_{r_n}x_n - p\|^2 \\
&= \|(F_{r_n}G_{r_n}x_n - \xi BF_{r_n}G_{r_n}x_n) - (p-\xi Bp)\|^2 \\
&= \frac{1}{2}[\|(F_{r_n}G_{r_n}x_n - \xi BF_{r_n}G_{r_n}x_n) - (p-\xi Bp)\|^2 \\
&\quad + \|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 \\
&\quad - \|((F_{r_n}G_{r_n}x_n - \xi BF_{r_n}G_{r_n}x_n) - (p-\xi Bp)) \\
&\quad - ((I-\xi B)F_{r_n}G_{r_n}x_n - p)\|^2] \\
&\leq \frac{1}{2}[\|F_{r_n}G_{r_n}x_n - p\|^2 + \|(I-\xi B)F_{r_n}G_{r_n}x_n - p)\|^2 \\
&\quad - \|(F_{r_n}G_{r_n}x_n - p) - ((I-\xi B)F_{r_n}G_{r_n}x_n - p)\|^2 \\
&\quad + 2\xi \langle F_{r_n}G_{r_n}x_n - (I-\xi B)\times F_{r_n}G_{r_n}x_n, BF_{r_n}G_{r_n}x_n\rangle \\
&\quad - \xi^2 \|BF_{r_n}G_{r_n}x_n\|^2]
\end{aligned}$$

which implies that

$$\begin{aligned}
\|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 &\leq \|x_n - p\|^2 - \|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\|^2 \\
&\quad + 2\xi \|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| \\
&\quad \times \|BF_{r_n}G_{r_n}x_n\| - \xi^2 \|BF_{r_n}G_{r_n}x_n\|^2 \qquad (3.14) \\
&\leq \|x_n - x_{n+1}\|^2 + 2\|x_{n+1}-p\| \times \|x_{n+1}-x_n\| \\
&\quad + \|x_{n+1}-p\|^2 - \|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\|^2 \\
&\quad + 2\xi \|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| \times \|BF_{r_n}G_{r_n}x_n\|.
\end{aligned}$$

But from the first line of inequality (3.10), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 \\
&\quad + \frac{\beta_n}{1-\beta_n}(2\|x_{n+1}-p\| + \|x_{n+1}-x_n\|) \times \|x_{n+1}-x_n\| \\
&\quad + \frac{2\alpha_n}{1-\beta_n} \times \langle \gamma g(x_n) - AW_n(I-\xi B)F_{r_n}G_{r_n}x_n, x_{n+1}-p\rangle.
\end{aligned}$$

So, for some real constant $M_5 > 0$, we obtain

$$\|x_{n+1}-p\|^2 \leq \|(I-\xi B)F_{r_n}G_{r_n}x_n - p\|^2 + (\alpha_n + \|x_{n+1}-x_n\|)M_5. \tag{3.15}$$

Now, using (3.15) in (3.14)and rearranging the terms, we obtain (for some real constants $M_6 > 0$) that

$$\|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\|^2 \leq (\alpha_n + \|x_{n+1}-x_n\| + \|BF_{r_n}G_{r_n}x_n\|)M_6$$

which implies that

$$\lim_{n\to\infty}\|F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| = 0.$$

But,

$$\begin{aligned}\|(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\| &\leq \|(I-\xi B)F_{r_n}G_{r_n}x_n - F_{r_n}G_{r_n}x_n\| \\ &\quad + \|F_{r_n}G_{r_n}x_n - x_n\| \to 0 \; as \; n\to\infty\end{aligned}$$

and

$$\begin{aligned}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| &\leq \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\| \\ &\quad + \|x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\|.\end{aligned}$$

The last inequality implies

$$\lim_{n\to\infty}\|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - (I-\xi B)F_{r_n}G_{r_n}x_n\| = 0.$$

We note that, since every real Hilbert space H is a uniformly convex real Banach space and $\{x_n\}_{n\geq 1}$ is a bounded sequence in H, then we obtain by Lemma 2.2 that there exists a unique $x^* \in H$ such that x^* is the asymptotic center of the sequence $\{x_{n_j}\}_{j\geq 1}$, (where $\{x_{n_j}\}_{j\geq 1}$ is a subsequence of $\{x_n\}_{n\geq 1}$).

STEP 5: In this step, we show that the asymptotic center of $\{x_n\}_{n\geq 1}$ belongs to Ω. Recall that $\lim\limits_{n\to\infty} r_n = r_0 > 0$ and by Lemma 2.3, $Wx = \lim\limits_{n\to\infty} W_n x$ exists for all $x \in H$. Furthermore, we have by Lemma 2.4 that $Fix(W) = \cap_{n=1}^{\infty} Fix(W_n) = \cap_{j=1}^{\infty} Fix(T_j)$. Thus, if x^* is the asymptotic center of $\{x_n\}_{n\geq 1}$, then

$$\begin{aligned}\|x_{n_j} - Wx^*\| &\leq \|x_{n_j+1} - x_{n_j}\| + \|\alpha_{n_j}\gamma g(x_{n_j}) + \beta_{n_j}x_{n_j} + ((1-\beta_{n_j})I - \alpha_{n_j}A) \\ &\quad \times W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - Wx^*\| \\ &\leq \|x_{n_j+1} - x_{n_j}\| + \beta_{n_j}\|x_{n_j} - Wx^*\| + (1-\beta_{n_j}) \\ &\quad \times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - W_{n_j}x_{n_j}\| + \|W_{n_j}x_{n_j} - W_{n_j}x^*\|\end{aligned}$$

$$+\|W_{n_j}x^* - Wx^*\|\Big) + \alpha_{n_j}\|\gamma g(x_{n_j}) - AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|$$
$$\leq \|x_{n_j+1} - x_{n_j}\| + \beta_{n_j}\|x_{n_j} - Wx^*\|$$
$$+(1\text{-}\beta_{n_j})\Big(\|(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - x_{n_j}\| + \|x_{n_j} - x^*\|$$
$$+\|W_{n_j}x^* - Wx^*\|\Big) + \alpha_{n_j}\|\gamma g(x_{n_j}) - AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.$$ Thus,

we obtain

$$\begin{aligned}\|x_{n_j} - Wx^*\| \quad &\leq \quad \frac{1}{1-\beta_{n_j}}\|x_{n_j+1} - x_{n_j}\| + \|(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - x_{n_j}\| + \|x_{n_j} - x^*\| \\ &\quad +\|W_{n_j}x^* - Wx^*\| + \frac{\alpha_{n_j}}{1-\beta_{n_j}}\|\gamma g(x_{n_j}) - AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

So that

$$\limsup_{j\to\infty}\|x_{n_j} - Wx^*\| \leq \limsup_{j\to\infty}\|x_{n_j} - x^*\|.$$

This implies by Proposition 2.1 (with $C = H$) and uniqueness of x^* that $Wx^* = x^*$, that is $x^* \in \bigcap_{j=1}^{\infty} Fix(T_j)$. Next,

$$\begin{aligned}\|x_{n_j} - F_{r_0}x^*\| \quad &\leq \quad \|x_{n_j+1} - x_{n_j}\| + \|\alpha_{n_j}\gamma g(x_{n_j}) + \beta_{n_j}x_{n_j} + [(1-\beta_{n_j})I - \alpha_{n_j}A] \\ &\quad \times W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - F_{r_0}x^*\| \\ &\leq \quad \|x_{n_j+1} - x_{n_j}\| + \beta_{n_j}\|x_{n_j} - F_{r_0}x^*\| + (1-\beta_{n_j}) \\ &\quad \times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - F_{r_{n_j}}x_{n_j}\| + \|F_{r_{n_j}}x_{n_j} - F_{r_{n_j}}x^*\| \\ &\quad +\|F_{r_{n_j}}x^* - F_{r_0}x^*\|\Big) + \alpha_{n_j}\|\gamma g(x_{n_j}) - AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\leq \quad \|x_{n_j+1} - x_{n_j}\| + \beta_{n_j}\|x_{n_j} - F_{r_0}x^*\| + (1-\beta_{n_j}) \\ &\quad \times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} - F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|G_{r_{n_j}}x_{n_j} - x_{n_j}\| + \|x_{n_j} - x^*\| + \frac{|r_{n_j} - r_0|}{r_0}\Big[\|F_{r_0}x^*\| + \|x^*\|\Big]\Big) \\ &\quad +\alpha_{n_j}\|\gamma g(x_{n_j}) - AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

So that we obtain

$$\begin{aligned}\|x_{n_j}-F_{r_0}x^*\| &\le \frac{1}{1-\beta_{n_j}}\|x_{n_j+1}-x_{n_j}\|+\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|G_{r_{n_j}}x_{n_j}-x_{n_j}\|+\|x_{n_j}-x^*\|+\frac{|r_{n_j}-r_0|}{r_0}\Big[\|F_{r_0}x^*\|+\|x^*\|\Big] \\ &\quad +\frac{\alpha_{n_j}}{1-\beta_{n_j}}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\|x_{n_j}-F_{r_0}x^*\|\le\limsup_{n\to\infty}\|x_{n_j}-x^*\|$$

So, by Proposition 2.1, we obtain that $F_{r_0}x^*=x^*$ which means $T'x^*=x^*$, that is, $x^*\in\mathrm{Fix}(T')$. Moreover,

$$\begin{aligned}\|x_{n_j}-G_{r_0}x^*\| &\le \|x_{n_j+1}-x_{n_j}\|+\|\alpha_{n_j}\gamma g(x_{n_j})+\beta_{n_j}x_{n_j}+[(1-\beta_{n_j})I-\alpha_{n_j}A] \\ &\quad \times W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-G_{r_0}x^*\| \\ &\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-G_{r_0}x^*\|+(1-\beta_{n_j}) \\ &\quad \times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-x_{n_j}\|+\|x_{n_j}-G_{r_{n_j}}x_{n_j}\|+\|G_{r_{n_j}}x_{n_j}-G_{r_{n_j}}x^*\| \\ &\quad +\|G_{r_{n_j}}x^*-G_{r_0}x^*\|\Big)+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-G_{r_0}x^*\|+(1-\beta_{n_j}) \\ &\quad \times\Bigg(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-x_{n_j}\|+\|x_{n_j}-G_{r_{n_j}}x_{n_j}\|+\|x_{n_j}-G_{r_{n_j}}\| \\ &\quad +\|x_{n_j}-x^*\|+\frac{|r_{n_j}-r_0|}{r_0}\Big(\|G_{r_0}x^*\|+\|x^*\|\Big)\Bigg) \\ &\quad +\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|x_{n_j}-G_{r_0}x^*\| &\le \frac{1}{1-\beta_{n_j}}\|x_{n_j+1}-x_{n_j}\|+\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j} \\ &\quad -F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|+\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-x_{n_j}\|+\|x_{n_j}-G_{r_{n_j}}x_{n_j}\| \\ &\quad +\|x_{n_j}-x^*\|+\frac{|r_{n_j}-r_0|}{r_0}\Big(\|G_{r_0}x^*\|+\|x^*\|\Big) \\ &\quad +\frac{\alpha_{n_j}}{1-\beta_{n_j}}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

Thus,

$$\limsup_{j\to\infty}\|x_{n_j}-G_{r_0}x^*\|\le\limsup_{j\to\infty}\|x_{n_j}-x^*\|.$$

By Proposition 2.1 again we have $G_{r_0}x^*=x^*$. Hence, $x^*\in EP(f)$. Having shown that $G_{r_0}x^*=x^*=F_{r_0}x^*$, using this fact and Lemma 2.13 we have

$$\begin{aligned}
\|x_{n_j}-(x^*-\xi Bx^*)\| &\le \|x_{n_j+1}-x_{n_j}\|+\|\alpha_{n_j}\gamma g(x_{n_j})+\beta_{n_j}x_{n_j}+[(1-\beta_{n_j})I-\alpha_{n_j}A]\\
&\quad\times W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(x^*-\xi Bx^*)\|\\
&\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\
&\quad\times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\quad+\|(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(x^*-\xi Bx^*)\|\Big)\\
&\quad+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\
&\quad\times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\quad+\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-x^*\|\Big)\\
&\quad+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\
&\quad\times\Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\quad+\|F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-F_{r_{n_j}}G_{r_0}x^*\|+\|F_{r_{n_j}}G_{r_0}x^*-F_{r_0}G_{r_0}x^*\|\Big)\\
&\quad+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\le \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\
&\quad\times\Big(W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\
&\quad+\|G_{r_{n_j}}x_{n_j}-G_{r_0}x^*\|+\|F_{r_{n_j}}G_{r_0}x^*-F_{r_0}G_{r_0}x^*\|\Big)\\
&\quad+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|,
\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n_j}-(x^*-\xi Bx^*)\| &\leq \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\ &\quad \Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\ &\quad +\|G_{r_{n_j}}x_{n_j}-G_{r_{n_j}}x^*\|+\|G_{r_{n_j}}x^*-G_{r_0}x^*\|+\|F_{r_{n_j}}G_{r_0}x^*\\ &\quad -F_{r_0}G_{r_0}x^*\|\Big)+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\ &\leq \|x_{n_j+1}-x_{n_j}\|+\beta_{n_j}\|x_{n_j}-(x^*-\xi Bx^*)\|+(1-\beta_{n_j})\\ &\quad \Big(\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\ &\quad +\|x_{n_j}-x^*\|+\frac{|r_{n_j}-r_0|}{r_0}\times(\|F_{r_0}G_{r_0}x^*\|\\ &\quad +2\|G_{r_0}x^*\|+\|x^*\|)\Big)+\alpha_{n_j}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|,\end{aligned}$$

which gives

$$\begin{aligned}\|x_{n_j}-(x^*-\xi Bx^*)\| &\leq \frac{1}{1-\beta_{n_j}}\|x_{n_j+1}-x_{n_j}\|\\ &\quad +\|W_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}-(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|\\ &\quad +\|x_{n_j}-x^*\|+4\|x^*\|\frac{|r_{n_j}-r_0|}{r_0}\\ &\quad +\frac{\alpha_{n_j}}{1-\beta_{n_j}}\|\gamma g(x_{n_j})-AW_{n_j}(I-\xi B)F_{r_{n_j}}G_{r_{n_j}}x_{n_j}\|.\end{aligned}$$

Thus, we have

$$\limsup_{j\to\infty}\|x_{n_j}-(x^*-\xi Bx^*)\|\leq \limsup_{j\to\infty}\|x_{n_j}-x^*\|$$

So that Proposition 2.1 gives $x^*-\xi Bx^*=x^*$ and this implies $x^*\in B^{-1}(0)$. Hence, $x^*\in\Omega$. **STEP 6:** Here, we show that if $x'\in\Omega$ is

the unique solution of the variational inequality (3.2), then

$$\limsup_{n\to\infty}\langle\gamma g(x')-Ax',x_n-x'\rangle\leq 0.$$

Now let $\{x_{n_j}\}_{j\geq 1}$ be the subsequence of $\{x_n\}_{n\geq 1}$ such that

$$\limsup_{n\to\infty}\langle\gamma g(x_n)-Ax',x_n-x'\rangle=\lim_{j\to\infty}\langle\gamma g(x_{n_j})-Ax',x_{n_j}-x'\rangle.$$

Let $t\in(0,1), x\in H$ be arbitrary and x^* be the asymptotic center of $\{x_{n_j}\}_{j\geq 1}$, where $\{x_{n_j}\}_{j\geq 1}$ is as given by (3.1), then using Lemma 2.5

we have

$$\begin{aligned}\|x_{n_j}-x^*-t(x-x^*)\|^2 &\leq \|x_{n_j}-x^*\|^2-2t\langle x-x^*,x_{n_j}-x^*-t(x-x^*)\rangle\\ &= \|x_{n_j}-x^*\|^2+2t\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle.\end{aligned}$$

Since x^* is the asymptotic center of $\{x_{n_j}\}_{j\geq 1}$, then using Proposition 2.1, we see that

$$\begin{aligned}\limsup_{j\to\infty}\|x_{n_j}-x^*\|^2 &\leq \limsup_{j\to\infty}\|x_{n_j}-x^*-t(x-x^*)\|^2\\ &\leq \limsup_{j\to\infty}\|x_{n_j}-x^*\|^2\\ &\quad +2t\limsup_{j\to\infty}\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle.\end{aligned}$$

Thus,

$$0\leq \limsup_{j\to\infty}\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle.$$

But,

$$\langle x-x^*,x^*-x_{n_j}\rangle=\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle-t\|x-x^*\|^2.$$

So,

$$\begin{aligned}\limsup_{j\to\infty}\langle x-x^*,x^*-x_{n_j}\rangle &= \limsup_{j\to\infty}(\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle-t\|x-x^*\|^2)\\ &= \limsup_{j\to\infty}\langle x-x^*,x^*+t(x-x^*)-x_{n_j}\rangle-t\|x-x^*\|^2\\ &\geq -t\|x-x^*\|^2.\end{aligned}$$

Since $t\in(0,1)$ is arbitrary, we obtain

$$\limsup_{j\to\infty}\langle x-x^*,x^*-x_{n_j}\rangle\geq 0\ \forall\, x\in H$$

In particular, for $x=x^*+\gamma g(x')-Ax'\in H$, also using the fact that $x^*\in\Omega$, we obtain

$$\begin{aligned}0 &\leq \limsup_{j\to\infty}\langle\gamma g(x')-Ax',x^*-x_{n_j}\rangle\\ &\leq \langle\gamma g(x')-Ax',x^*-x'\rangle+\limsup_{j\to\infty}\langle\gamma g(x')-Ax',x'-x_{n_j}\rangle\\ &= \langle\gamma g(x')-Ax',x^*-x'\rangle+\lim_{j\to\infty}\langle\gamma g(x')-Ax',x'-x_{n_j}\rangle\\ &\leq \lim_{j\to\infty}\langle\gamma g(x')-Ax',x'-x_{n_j}\rangle\end{aligned}$$

which implies that

$$\lim_{j\to\infty}\langle\gamma g(x')-Ax',x_{n_j}-x'\rangle\leq 0.$$

Hence,

$$\limsup_{n\to\infty}\langle \gamma g(x') - Ax', x_n - x'\rangle = \lim_{j\to\infty}\langle \gamma g(x') - Ax', x_{n_j} - x'\rangle \le 0.$$

STEP 7: Finally, we show that $\{x_n\}_{n\ge 1}$ converges strongly to x', the unique solution of (3.2). Observe that from the iterative sequence (3.1) and Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1} - x'\|^2 &= \|\alpha_n \gamma g(x_n) + [(1-\beta_n)I - \alpha_n A] \times W_n(I-\xi B)F_{r_n}G_{r_n}x_n + \beta_n x_n - x'\|^2 \\
&= \|[(1-\beta_n)I - \alpha_n A] \times (W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x') + \beta_n(x_n - x') \\
&\quad + \alpha_n(\gamma g(x_n) - Ax')\|^2 \\
&\le \|[(1-\beta_n)I - \alpha_n A] \times (W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x') + \beta_n(x_n - x')\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Ax', x_{n+1} - x'\rangle \\
&\le (1-\beta_n - \alpha_n\gamma + \beta_n)^2\|x_n - x'\|^2 + 2\alpha_n^2\|\gamma g(x_n) - Ax'\|^2 + 2\alpha_n(1-\beta_n) \\
&\quad \times\langle \gamma g(x_n) - Ax', W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\rangle \\
&\quad + 2\alpha_n(1-\beta_n)\langle \gamma g(x_n) - Ax', x_n - x'\rangle + 2\beta_n\alpha_n\langle u - Ax', x_n - x'\rangle \\
&\quad - 2\alpha_n^2\langle \gamma g(x_n) - Ax', A(W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x')\rangle \\
&= [1-2\alpha_n\gamma]\|x_n - x'\|^2 + \alpha_n^2\gamma^2\|x_n - x'\|^2 + 2\alpha_n^2\|\gamma g(x_n) - Ax'\|^2 \\
&\quad + 2\alpha_n(1-\beta_n) \times \langle \gamma g(x_n) - Ax', W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\rangle \\
&\quad - 2\alpha_n^2\langle \gamma g(x_n) - Ax', AW_n(I-\xi B)F_{r_n}G_{r_n}x_n - x'\rangle \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Ax', x_n - x'\rangle \\
&\le [1-2\alpha_n\gamma]\|x_n - x'\|^2 + \alpha_n(\alpha_n + \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\|)M_7 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Ax', x_n - x'\rangle \\
&= [1-2\alpha_n\gamma]\|x_n - x'\|^2 + \alpha_n(\alpha_n + \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\|)M_7 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - \gamma g(x'), x_n - x'\rangle + 2\alpha_n\langle \gamma g(x') - Ax', x_n - x'\rangle \\
&\le [1-2\alpha_n\gamma]\|x_n - x'\|^2 + \alpha_n(\alpha_n + \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\|)M_7 \\
&\quad + 2\alpha_n\gamma k\|x_n - x'\|^2 + 2\alpha_n\langle \gamma g(x') - Ax', x_n - x'\rangle
\end{aligned}$$

$$\begin{aligned}
= \;& [1-2\alpha_n\gamma(1-k)]\|x_n - x'\|^2 + \alpha_n(\alpha_n + \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\|)M_7 \\
& + 2\alpha_n\langle \gamma g(x') - Ax', x_n - x'\rangle,
\end{aligned}$$

for some real constant $M_7 > 0$. Thus,

$$\|x_{n+1} - x'\|^2 \le (1-\delta_n)\|x_n - x'\|^2 + \sigma_n,$$

where $\delta_n = 2\alpha_n\gamma(1-k)$ and $\sigma_n = \alpha_n(\alpha_n + \|W_n(I-\xi B)F_{r_n}G_{r_n}x_n - x_n\|)M_6 + 2\alpha_n\langle \gamma g(x') - Ax', x_n - x'\rangle$. But using 3.7, Step 6 and the fact that $\lim_{n\to\infty}\alpha_n = 0$, we see that $\limsup_{n\to\infty}(\sigma_n/\delta_n) \le 0$. Hence, we obtain by Lemma 2.7 that the sequence $\{x_n\}_{n\geq 1}$ converges strongly to a unique solution $x' \in \Omega$ of (3.2). This completes the proof. $\square$

It is worth to note that Marino and Xu [21] pointed out that the unique solution $x' \in F(T)$ in Theorem 1.2 is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \left(\frac{1}{2}\langle Ax, x\rangle - h(x)\right), \tag{3.16}$$

where h is a potential function for γg (that is, $h'(x) = \gamma g(x)$ for all $x \in H$). Thus, the unique solution approximated by the iteratives algorithm introduced and studied in this paper does not only solve the variational inequality problem

$$\left\langle (\gamma g - A)x', y - x' \right\rangle \leq 0,\ y \in F(T),$$

but it is also the optimality condition of the minimization problem (3.16).

Algorithm (3.1) is more encompassing and more general than algorithms (1.2), (1.3), (1.4), (1.6), (1.7) and (1.8) in the sense that (3.1) is a model for finding approximate solutions of more number of problems when compared with the others. Thus, Theorem 3.1 extends, generalizes, improves and unifies the corresponding results of Yamada [34], Marino and Xu [21], Tian [30], Yao *et al.* [35], Colao and Marino [14], Chamnarnpan and Kumam [10]. Moreover, the problem of well-defindness observed in [10] is taken care of. Furthermore, it is worthy to note that there is no further generalization in considering the so called *generalized mixed equilibrium problem* (see, for example, [1, 2, 13, 23]) instead of equilibrium problem considered in this paper since methods employed in handling both are virtually the same; moreover, it could easily be shown that none of them is a generalization of the other.

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