Journal of the Nigerian Mathematical Society
Vol. 43, Issue 1, pp. 15 - 28, 2024

COMPARISON OF BLOCK IMPLICIT ALGORITHMS AND RUNGE KUTTA METHODS FOR THE SOLUTION OF NON LINEAR FIRST ORDER PROBLEMS WITH LEGENDRE POLYNOMIAL BASIC FUNCTIONS

A. M. BADMUS [1], A. O. SUBAIR [2]

ABSTRACT. A family of uniform and non-uniform order Linear Multistep Block Methods was developed using Legendre Polynomial as the basic functions and reconstructed to its equivalent Runge – Kutta type Methods. The implicit block method at $k = 3$ gives a uniform order 6 while implicit block method at $k = 4$ gives non uniform order $6 \leq x \leq 9$. The continuous formulation of the method were evaluated at some grid and off grid points to obtain the implicit block methods. Also both methods were demonstrated on non-linear first order initial value problems (IVPs) and the results obtained compared favorably with the analytic solution.

1. INTRODUCTION

Most life and physical problems arising from engineering, biology, mathematics, physics, and many other branches of science, are naturally non-linear in nature. This paper concerned with the derivation and the analysis of numerical solutions for non-linear first order Ordinary Differential Equations (ODEs) of the form.

$$y' = f(x,y), \qquad y(x_0) = y_0 \text{ for } a \leq x \leq b \tag{1.1}$$

where $y' \equiv \frac{dy}{dx}$ and f is a given real valued function of two variables. Since most non-linear differential equations do not have close form solutions. Hence there is a great need for suitable numerical computations to handle such class of problems.

Many numerical techniques are available for the solution of (1.1) such

Received by the editors October 12, 2022; Revised: October 24, 2023; Accepted: February 12, 2024

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp. it/jnms/

2010 *Mathematics Subject Classification.* 65H11, 65K05, 65H12, 65H18.

Key words and phrases. Block linear multi-step methods, Implicit Block Method, Runge- Kutta method, Uniform and non-uniform Order, Zero Stable.

as Adams-Moulton, Runge Kutta, Euler's rule etc. they all have their permanent characteristic advantages and disadvantages. The Euler's method is known explicit one step method and it requires no additional starting values. Its low order makes it of low practical value. Linear multi-step method (LMM) achieves higher order with respect to y_{n+j} , f_{n+j} Many authors have worked extensively on Block linear Multi-step methods among them are [1], [2],[3],[4] and [6] worked on some implicit K-step hybrid block methods for solutions of (1.1) using power series as the basic functions. The aim of this paper is to derive some implicit block LMM using Legendre polynomial as basis (basic) functions, reformulate them into equivalent Runge – Kutta type methods and use both methods to obtain the numerical solution of non-linear problems of first order odes

Linear Multi-step method

Definition 1.0 Linear Multi-step method: A LMM with k-step size have the form;

$$y_n = \sum_{j=0}^{k} \alpha_j y_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

where α_j and β_j are constants, y_n is the numerical solution at $x = x_n$, $f_n = f(x_n, y_n)$. If $\beta_k \neq 0$, the LMM becomes implicit scheme, otherwise explicit [8]

Definition 1.1 Zero Stability: The LMM (1.2) is said to satisfy the root conditions if all the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple.

Definition 1.2 Runge - Kutta: This method is a family of implicit and explicit iterative methods used in temporal discretization for the approximate solution of ODEs. They are single step methods however, with multistep stages per step. They do not require derivatives or the right hand side functions f in the code, and are therefore general purpose IVPs solvers. The Runge-Kutta integration method for (1.1) is given by;

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j k_j$$

where

$$k_i = f(x + c_i, y + \sum_{j=1}^{s} b_j k_j) \quad i = 1, 2, \cdots$$

This method can be expressed in a tableau form (Butcher tableau) as follows

$$\begin{array}{c|ccccc} C & & & A & & \\ \hline c_1 & a_{11} & a_{12} & \cdots & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & \cdots & a_{ss} \\ \hline b^T & b_1 & b_2 & \cdots & \cdots & b_s \end{array}$$

where $A = (a_{ij}),\ i = 1,2,\cdots,s$ is an $S \times S$ Matrix, $b^T = b_1, b_2, \cdots, b_s$ and

$$C_i = \sum_{j=0}^{s-1} a_{ij}$$

.

2. Legendre polynomial

The Legendre polynomials are obtained from expansion of a single cosine rule for triangles or from a solution of Legendre's differential equation. The starting point is by differentiating

$$G(_{z,t}) = \frac{1}{\sqrt{1-t^2-2zt}} = \sum_{n=0}^{\infty} p_n(z)t^n$$

with respect to t and after extracting the coefficients we used some assumptions to produce

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n \tag{2.1}$$

$$P'n(z) = \frac{2nz}{2^n n!}\frac{d^n}{dx^n}(z^2-1)^{n-1}$$

The (2.1) is known as the Rodrigues formula which is used to generate the basis as

$$P_0(z) = 1$$

$$P_1(z) = z$$

$$P_2(z) = \frac{1}{2}(3z^2-1)$$

$$P_3(z) = \frac{1}{2}(5z^3-3z)$$

$$P_4(z) = \frac{1}{8}(35z^4-30z^2+3)$$

$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$$
$$P_6(z) = \frac{1}{16}(231z^6 - 315z^4 + 105z^2 - 5)$$
$$P_7(z) = \frac{1}{16}(429z^7 - 693z^5 + 315z^3 - 3z5)$$
$$\vdots$$
$$\vdots$$

3. DERIVATIVE OF THE METHOD

For derivation of block method at $k = 3$, given a Legendre polynomial series of the form

$$y(x) = \sum_{j=0}^{p+c-1} \alpha_j P_j(z) = y_{n+j} \tag{3.1}$$

$$y\prime(x) = \sum_{j=1}^{p+c-1} \beta_j P\prime_j(z) = f_{n+j} \tag{3.2}$$

(3.1)is interpolated at $x = x_{n+j}, j = (0,1,2)$ and collocate (3.2) at $j = (\frac{3}{2}, 2, \frac{5}{2}, 3)$ to form our non linear equations of the form

$$\sum_{j=0}^{p+c-1} \alpha_j P_j(z) = y_{n+j} \quad , j = 0,1,2 \tag{3.3}$$

$$\sum_{j=1}^{p+c-1} \beta_j P\prime_j(z) = f_{n+j} \quad , j = (\frac{3}{2}, 2, \frac{5}{2}, 3) \tag{3.4}$$

When using Maple 17 mathematical software to obtain the values of α_j and β_j in (3.3) and (3.4) and substituting the values into (3.5) to obtain our continuous formulation as

$$y_x = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h[\beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3}] \tag{3.5}$$

where

$$\alpha_0 = \left\{ \frac{1}{67844} \frac{67844h^6 + 111951h^4 + 19257h^2 + 232}{h^6} - \frac{3}{33922} \frac{(39690h^5 + 24381h^3 + 1404h)x}{h^5} \right.$$
$$+ \frac{1}{101766} \frac{(33583h^4 + 82530h^2 + 1160)(\frac{3}{2}x^2 - \frac{1}{2})}{h^5}$$

$$-\frac{1}{7269}\frac{(10449h^3+936h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}+\frac{6}{186571}\frac{(10087h^2+232)(\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8})}{h^5}$$
$$\left.-\frac{624}{16961}\frac{\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x}{h^4}+\frac{928}{559713}\frac{\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}}{h^5}\right\}$$

$$\alpha_1=\left\{\frac{-1}{16961}\frac{569268h^4+152355h^2+2216}{h^6}+\frac{12}{16961}\frac{(57960h^5+81641h^3+6204h)x}{h^5}\right.$$
$$-\frac{4}{50883}\frac{(853902h^4+326475h^2+5540)(\frac{3}{2}x^2-\frac{1}{2})}{h^5}+\frac{8}{7269}\frac{(34989h^3+4136h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}$$
$$-\frac{24}{186571}\frac{(79805h^2+2216)(\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8})}{h^5}+\frac{66176}{50883}\frac{\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x}{h^4}$$
$$\left.-\frac{35456}{559713}\frac{\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}}{h^5}\right\}$$

$$\alpha_2=\left\{\frac{1}{67844}\frac{2165121h^4+590163h^2+8632}{h^6}-\frac{3}{33922}\frac{(423990h^5+628747h^3+48228h)x}{h^5}\right.$$
$$+\frac{1}{101766}\frac{(6495363h^4+2529270h^2+43160)(\frac{3}{2}x^2-\frac{1}{2})}{h^5}-\frac{1}{7269}\frac{(269463h^3+32152h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}$$
$$+\frac{6}{186571}\frac{(309133h^2+8632)(\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8})}{h^5}-\frac{64304}{50883}\frac{\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x}{h^4}$$
$$\left.+\frac{34528}{559713}\frac{\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}}{h^5}\right\}$$

$$\beta_{\frac{3}{2}}=\left\{\frac{-8}{109035}\frac{396080h^4+117723h^2+1518}{h^5}+\frac{8}{254415}\frac{(1035300h^5+1689975h^3+138735)x}{h^5}\right.$$
$$-\frac{16}{152649}\frac{(554512h^4+235446h^2+4235)(\frac{3}{2}x^2-\frac{1}{2})}{h^5}+\frac{16}{109035}\frac{(241425h^3+30830h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}$$
$$-\frac{64}{254415}\frac{(39241h^2+1155)(\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8})}{h^5}+\frac{197312}{152649}\frac{\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x}{h^5}$$
$$\left.-\frac{1408}{21807}\frac{\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}}{h^5}\right\}$$

$$\beta_2=\left\{\frac{-1}{73690}\frac{-77535h^4-90993h^2-2280}{h^5}+\frac{1}{84805}\frac{(33075h^5-412188h^3-71520h)x}{h^5}\right.$$
$$+\frac{1}{50883}\frac{(108549h^4+181986h^2+5320)(\frac{3}{2}x^2-\frac{1}{2})}{h^5}-\frac{4}{109035}\frac{(88326h^3+23840h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}$$
$$+\frac{4}{932855}\frac{(333641h^2+15960)(\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8})}{h^5}-\frac{38144}{152649}\frac{\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x}{h^4}$$

$$+\frac{1216}{79959}\frac{\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}}{h^5}\Bigg\}$$

$$\beta_{\frac{5}{2}}=\Bigg\{\frac{-8}{254415}\frac{88200h^4+40341h^2+885}{h^5}+\frac{8}{84805}\frac{(26460h^5+67137h^3+8205h)x}{h^5}$$
$$-\frac{16}{50883}\frac{(17640h^4+11526h^2295)(\frac{3}{2}x^2-\frac{1}{2})}{h^5}+\frac{16}{109035}\frac{(28773h^3+5470h)(\frac{5}{2}x^3-\frac{3}{2}x)}{h^5}$$

Evaluating (3.5) at $x=x_{n+j}, j=\frac{3}{2},\frac{5}{2},3$ and it's first derivative evaluated at $x=x_{n+j}, j=0,1$ .We obtain the following five discrete schemes as our Block method.

$$\frac{281}{155072}y_n-\frac{5859}{38768}y_{n+1}-y_{n+\frac{3}{2}}+\frac{178227}{155072}y_{n+2}$$
$$=\frac{159}{38768}hf_{n+3}-\frac{675}{19384}hf_{n+\frac{5}{2}}+\frac{24111}{77536}hf_{n+2}+\frac{7119}{19384}hf_{n+\frac{3}{2}}$$
$$\frac{297}{155072}y_n-\frac{4675}{38768}y_{n+1}-\frac{173475}{155072}y_{n+2}-y_{n+\frac{5}{2}}$$
$$=\frac{375}{38768}hf_{n+3}-\frac{4335}{19384}hf_{n+\frac{5}{2}}-\frac{25425}{77536}hf_{n+2}+\frac{3075}{19384}hf_{n+\frac{3}{2}}$$
$$\frac{7}{2423}y_n-\frac{351}{2423}y_{n+1}-\frac{2079}{2423}y_{n+2}+y_{n+3}$$
$$=\frac{378}{2423}hf_{n+3}+\frac{1728}{2423}hf_{n+\frac{5}{2}}+\frac{270}{2423}hf_{n+2}+\frac{384}{2423}hf_{n+\frac{3}{2}}$$
$$8505y_n-99360y_{n+1}+90855y_{n+2}$$
$$=-2423hf_n+945hf_{n+2}-1100hf_{n+3}+78880hf_{n+\frac{3}{2}}+6048hf_{n+\frac{5}{2}}$$
$$13365y_{n+2}-45y_n-13320y_{n+1}$$
$$=2423hf_n+2418hf_{n+2}-7hf_{n+3}+8608hf_{n+\frac{3}{2}}-32hf_{n+\frac{5}{2}} \quad (3.6)$$

The order of the hybrid block method (3.6) at $k=3$ gives uniform order 6 with the following error constants

$$\left[-\frac{2379}{86840320},\frac{95}{2481152},-\frac{111}{1356880},-\frac{2103}{112},-\frac{41}{168}\right]^T$$

3.1. **Derivation of second method at $k = 4$.** Equation (3.1) is interpolated at $x = x_n$ and equation (3.2) is collocated at $x = x_{n+j}, j = (0, 1, \frac{2}{3}, 3, 4)$ to form Non-linear system of the form

$$\sum_{j=0}^{p+c-1} \alpha_j P_j(z) = y_{n+j}, \quad j = 0 \tag{3.7}$$

$$\sum_{j=0}^{p+c-1} \alpha_j P\prime_j(z) = f_{n+j}, \quad j = 0, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4 \tag{3.8}$$

The continuous formulation of (3.7) and (3.8) are of the form

$$y_x + \alpha_0 y_n = h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_4 f_{n+4}] \tag{3.9}$$

Following the same procedures used in (3.5) ,we obtain our continuous formulation and then evaluated at $x = x_{n+j},\ j = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$ to have the following six discrete schemes below

$$y_{n+1} - y_n = \frac{3923}{15120} hf_n + \frac{16483}{7560} hf_{n+1} - \frac{3124}{945} hf_{n+\frac{3}{2}} + \frac{221}{70} hf_n + 2$$
$$- \frac{1612}{945} hf_{n+\frac{5}{2}} + \frac{3253}{7560} hf_{n+3} - \frac{47}{3024} hf_{n+4}$$

$$y_{n+\frac{3}{2}} - y_n = \frac{463}{1792} hf_n + \frac{10611}{4480} hf_{n+1} - \frac{809}{280} hf_{n+\frac{3}{2}} + \frac{6723}{2240} hf_{n+2}$$
$$- \frac{459}{280} hf_{n+\frac{5}{2}} + \frac{1861}{4480} hf_{n+3} - \frac{27}{1792} hf_{n+4}$$

$$y_{n+2} - y_n = \frac{163}{630} hf_n + \frac{2222}{945} hf_{n+1} - \frac{832}{315} hf_{n+\frac{3}{2}} + \frac{116}{35} hf_{n+2}$$
$$- \frac{320}{189} hf_{n+\frac{5}{2}} + \frac{134}{315} hf_{n+3} - \frac{29}{315} hf_{n+4}$$

$$y_{n+\frac{5}{2}} - y_n = \frac{12505}{48384} hf_n + \frac{57125}{24192} hf_{n+1} - \frac{4075}{1512} hf_{n+\frac{3}{2}} + \frac{1625}{448} hf_{n+2}$$
$$- \frac{2185}{1512} hf_{n+\frac{5}{2}} + \frac{9875}{24192} hf_{n+3} - \frac{725}{48384} hf_{n+4}$$

$$y_{n+3}-y_n=\frac{29}{112}hf_n+\frac{659}{280}hf_{n+1}-\frac{92}{35}hf_{n+\frac{3}{2}}+\frac{243}{35}hf_{n+2}$$

$$-\frac{36}{35}hf_{n+\frac{5}{2}}+\frac{167}{280}hf_{n+3}-\frac{9}{560}hf_{n+4}$$

$$y_{n+4}-y_n=\frac{46}{189}hf_n+\frac{2624}{945}hf_{n+1}-\frac{4096}{945}hf_{n+\frac{3}{2}}+\frac{232}{35}hf_{n+2}$$

$$-\frac{4096}{945}hf_{n+\frac{5}{2}}+\frac{2624}{945}hf_{n+3}+\frac{46}{189}hf_{n+4}$$

(3.10)

The order of the hybrid block LMM (3.10) at $k=4$ gives non constant uniform order $[7,7,7,7,7,8]^T$ with error constant as

$$\left[-\frac{9}{8960},-\frac{225}{229376},-\frac{1}{1008},-\frac{225}{229376},-\frac{9}{8960},\frac{23}{56700}\right]^T$$

,

4. REFORMULATION INTO RUNGE-KUTTA TYPE METHOD

By Butcher array as:

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}$$

We reformulate equation(3.6) in the Butcher array as

C	A					
$\frac{1}{3}$	$\frac{11}{120}$	$\frac{673}{1080}$	$\frac{-104}{135}$	$\frac{211}{360}$	$\frac{-32}{135}$	$\frac{43}{1080}$
$\frac{1}{2}$	$\frac{35}{384}$	$\frac{441}{640}$	$\frac{-77}{120}$	$\frac{351}{640}$	$\frac{-9}{40}$	$\frac{73}{1920}$
$\frac{2}{3}$	$\frac{37}{405}$	$\frac{92}{135}$	$\frac{-224}{405}$	$\frac{29}{45}$	$\frac{-32}{135}$	$\frac{16}{405}$
$\frac{5}{6}$	$\frac{35}{384}$	$\frac{2375}{3456}$	$\frac{-125}{216}$	$\frac{875}{1152}$	$\frac{-35}{216}$	$\frac{125}{3456}$
1	$\frac{11}{120}$	$\frac{27}{40}$	$\frac{-8}{15}$	$\frac{27}{40}$	0	$\frac{11}{120}$
1	$\frac{11}{20}$	$\frac{27}{40}$	$\frac{-8}{15}$	$\frac{27}{40}$	0	$\frac{11}{120}$

We obtain an implicit 6 – Stage Runge – Kutta Type method for first order ODEs as follows

$$y_{n+1} = y_n + h\left[\left(\frac{11}{120}k_1 + \frac{27}{40}k_2 - \frac{8}{15}k_3 + \frac{27}{40}k_4 + 0k_5 + \frac{11}{120}k_6\right)\right] \tag{4.1}$$

where

$$k_1 = f\left[\left(x_n, y_n\right)\right]$$

$$k_2 = f\left[x_n + \frac{1}{3}h, y_n + h\left(\frac{11}{120}k_1 + \frac{673}{1080}k_2 - \frac{104}{135}k_3 + \frac{211}{360}k_4 + -\frac{32}{135}k_5 + \frac{43}{1080}k_6\right)\right]$$

$$k_3 = f\left[x_n + \frac{1}{2}h, y_n + h\left(\frac{35}{384}k_1 + \frac{441}{640}k_2 - \frac{77}{120}k_3 + \frac{351}{640}k_4 - \frac{9}{40}k_5 + \frac{73}{1920}k_6\right)\right]$$

$$k_4 = f\left[x_x + \frac{2}{3}h, y_n + h\left(\frac{37}{405}k_1 + \frac{92}{135}k_2 - \frac{224}{405}k_3 + \frac{29}{45}k_4 - \frac{32}{135}k_5 + \frac{16}{405}k_6\right)\right]$$

$$k_5 = f\left[x_n + \frac{5}{6}h, y_n + h\left(\frac{35}{384}k_1 + \frac{2375}{3456}k_2 - \frac{125}{216}k_3 + \frac{875}{1152}k_4 - \frac{35}{216}k_5 + \frac{125}{3456}k_6\right)\right]$$

$$k_6 = f\left[x_n + h, y_n + h\left(\frac{11}{120}k_1 + \frac{27}{40}k_2 - \frac{8}{15}k_3 + \frac{27}{40}k_4 + 0k_5 + \frac{11}{120}k_6\right)\right]$$

Also, following the same procedure to reformulate (3.10) of method $k=4$ into it equivalent Runge-Kutta type, as

$$y_{n+1} = y_n + h\left[\frac{23}{378}k_1 + \frac{656}{945}k_2 - \frac{1024}{945}k_3 + \frac{58}{35}k_4 - \frac{1024}{945}k_5 + \frac{656}{945}k_6 + \frac{23}{378}k_7\right] \quad (4.2)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left[x_n + \frac{1}{4}h, y_n + h\left(\frac{3923}{60480}k_1 + \frac{16483}{30240}k_2 - \frac{3124}{3780}k_3 + \frac{221}{280}k_4 - \frac{1612}{3780}k_6 - \frac{47}{12096}k_7\right)\right]$$

$$k_3 = f\left[x_n + \frac{3}{8}h, y_n + h\left(\frac{463}{d7168}k_1 + \frac{10611}{17920}k_2 - \frac{809}{1120}k_3 + \frac{6723}{8960}k_4 - \frac{459}{1120}k_5 + \frac{1861}{17920}k_6 - \frac{27}{7168}k_7\right)\right]$$

$$k_4 = f\left[x_n + \frac{1}{2}h, y_n + h\left(\frac{163}{2520}k_1 + \frac{2222}{3780}k_2 - \frac{832}{1260}k_3 + \frac{116}{140}k_4 - \frac{320}{756}k_5 + \frac{134}{1260}k_6 - \frac{29}{7560}k_7\right)\right]$$

$$k_5 = f\left[x_n + \frac{5}{8}h, y_n + h\left(\frac{12505}{193536}k_1 + \frac{57125}{96768}k_2 - \frac{4075}{6048}k_3 + \frac{1625}{1792}k_4 - \frac{2185}{6048}k_5 + \frac{9875}{96786}k_6 - \frac{725}{193536}k_7\right)\right]$$

$$k_6 = f\left[x_n + \frac{1}{3}h, y_n + h\left(\frac{29}{448}k_1 + \frac{657}{1120}k_2 - \frac{92}{140}k_3 + \frac{243}{280}k_4 - \frac{36}{140}k_5 + \frac{167}{1120}k_6 - \frac{0}{2240}k_7\right)\right]$$

$$k_7 = f\left[x_n + h, y_n + h\left(\frac{23}{378}k_1 + \frac{656}{945}k_2 - \frac{1024}{945}k_3 + \frac{58}{35}k_4 - \frac{1024}{945}k_5 + \frac{656}{945}k_6 + \frac{23}{378}k_7\right)\right]$$

5. Analysis of the methods

The necessary and sufficient conditions for LMM to be convergent are that it must be consistent and zero stable (Which is the fundamental theorem of Dhalquist) [7].

Block method at $k=3$ in (3.6) is arranged and defined in matrix form as

$$\begin{bmatrix} \frac{-5859}{38768} & -1 & \frac{178227}{155072} & 0 & 0 \\ \frac{-4675}{38768} & 0 & \frac{6939}{6203} & -1 & 0 \\ \frac{-351}{2423} & 0 & -\frac{20179}{2423} & 0 & 1 \\ \frac{-351}{2423} & 0 & -\frac{2079}{2423} & 0 & 0 \\ -99360 & 0 & 90855 & 0 & 0 \\ -13320 & 0 & 13365 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{281}{155072} \\ 0 & 0 & 0 & 0 & -\frac{297}{155072} \\ 0 & 0 & 0 & 0 & -\frac{7}{2423} \\ 0 & 0 & 0 & 0 & -8505 \\ 0 & 0 & 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n_1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{7119}{19384} & \frac{24111}{77536} & -\frac{675}{19384} & \frac{159}{38768} \\ 0 & \frac{3075}{19384} & -\frac{25425}{77536} & -\frac{4335}{19384} & \frac{375}{38768} \\ 0 & \frac{384}{2423} & \frac{270}{2423} & \frac{1728}{2423} & \frac{378}{2423} \\ 0 & 78880 & 945 & 6048 & -1100 \\ 2423 & 8608 & 2418 & -32 & -7 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2424 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n_2} \\ f_{n_{\frac{3}{2}}} \\ f_{n_1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \qquad (5.1)$$

where;

$$A^{(0)} = \begin{bmatrix} -\frac{5859}{38768} & -1 & \frac{178227}{155072} & 0 & 0 \\ -\frac{4675}{38768} & 0 & \frac{6939}{6203} & -1 & 0 \\ -\frac{351}{2423} & 0 & -\frac{2079}{2423} & 0 & 1 \\ -99369 & 0 & 90855 & 0 & 0 \\ 13320 & 0 & 13365 & 0 & 0 \end{bmatrix}.$$

Then

$$A^{(0)^{-1}} = \begin{bmatrix} 0 & 0 & 0 & -\frac{11}{96920} & \frac{673}{872280} \\ -1 & 0 & 0 & -\frac{35}{310144} & \frac{1323}{1550720} \\ 0 & 0 & 0 & -\frac{37}{327105} & \frac{92}{109035} \\ 0 & -1 & 0 & -\frac{35}{310144} & \frac{2375}{2791296} \\ 0 & 0 & 1 & -\frac{11}{96920} & \frac{81}{96920} \end{bmatrix}$$

Now normalized (5.1) by multiplying $(A^0)^{-1}$ to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+2} \\ y_{n+\frac{3}{2}} \\ y_{n+1} \\ y_{n+\frac{1}{2}} \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

$$+\begin{bmatrix} \frac{673}{1080} & -\frac{104}{135} & \frac{211}{360} & \frac{-32}{135} & \frac{43}{1080} \\ \frac{411}{640} & \frac{-77}{120} & \frac{351}{640} & \frac{-9}{40} & \frac{73}{1920} \\ \frac{92}{135} & \frac{-224}{405} & \frac{29}{45} & \frac{-32}{135} & \frac{16}{405} \\ \frac{2375}{3456} & \frac{-125}{216} & \frac{875}{1152} & \frac{-35}{216} & \frac{125}{3456} \\ \frac{27}{40} & \frac{-8}{15} & \frac{27}{40} & 0 & \frac{11}{120} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{32}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \tag{5.2}$$

From definition of zero stability (1.1),
Then

$$\rho(R) = \det\left[R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] = 0$$

This implies,

$$\rho(R) = \begin{bmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = R^5 - R^4 = 0$$

Which implies that $R_1 = R_2 = R_3 = R_4 = 0$ and $R_5 = 1$ and this implies the method is zero stable and consistent with order > 1.

Also following the same procedure to analyze (16) of method $K = 4$, we found out that $R_1 = R_2 = R_3 = R_4 = R_5 = 0$ and $R_6 = 1$; hence,both methods are zero stable and also consistent with order > 1

6.0 Numerical Experiments

Two non-linear first IVPs were used to ascertain the efficiency of our two methods

Problem 1.

$y' = 1 - y^2$

TABLE 1. Absolute Error for Problem 1

x	Method at k=3 (9)	LMM at K=3	LMM at K=4	R.K AT K=3	R.K AT K=4
0.1	3.60431×10^{-9}	3.6197×10^{-8}	1.0669×10^{-8}	2.3797×10^{-11}	1.1000×10^{-14}
0.2	1.72906×10^{-9}	3.4694×10^{-8}	1.0226×10^{-8}	4.0037×10^{-11}	2.2000×10^{-14}
0.3	6.6690×10^{-9}	3.4404×10^{-8}	9.8843×10^{-9}	4.4685×10^{-11}	2.9000×10^{-14}
0.4	2.94404×10^{-9}	1.0626×10^{-8}	2.2833×10^{-9}	3.8277×10^{-11}	3.1000×10^{-14}
0.5	3.31186×10^{-9}	1.0021×10^{-8}	2.2412×10^{-9}	2.4839×10^{-11}	2.7000×10^{-14}
0.6	2.99503×10^{-9}	7.8042×10^{-9}	2.0243×10^{-9}	9.5340×10^{-12}	1.9000×10^{-14}
0.7	1.27065×10^{-9}	1.2375×10^{-8}	1.8240×10^{-9}	3.4680×10^{-12}	1.1000×10^{-14}
0.8	1.93718×10^{-9}	1.0682×10^{-8}	6.9651×10^{-11}	1.1998×10^{-11}	1.0000×10^{-15}
0.9	3.07730×10^{-9}	1.0181×10^{-8}	2.7426×10^{-9}	1.5809×10^{-11}	7.0000×10^{-15}
1.0	5.77059×10^{-10}	8.9687×10^{-9}	2.3359×10^{-9}	1.5865×10^{-11}	1.1000×10^{-14}

TABLE 2. Absolute Error for Problem 2

X	LMM Method at k=3	LMM Method at k=4	R-K TYPE at k=3	R-K TYPE at k=4
0.1	3.1614×10^{-8}	2.7950×10^{-8}	1.6400×10^{-12}	2.0000×10^{-14}
0.2	3.1354×10^{-8}	2.7723×10^{-8}	1.6070×10^{-11}	4.0000×10^{-14}
0.3	3.4094×10^{-8}	2.8521×10^{-8}	6.0190×10^{-11}	7.0000×10^{-14}
0.4	5.5686×10^{-7}	8.4683×10^{-9}	1.6695×10^{-10}	1.5000×10^{-13}
0.5	5.7371×10^{-7}	8.5497×10^{-7}	4.1712×10^{-10}	3.3000×10^{-13}
0.6	6.4434×10^{-7}	9.0007×10^{-7}	1.0341×10^{-9}	7.3000×10^{-13}
0.7	1.1253×10^{-5}	1.0025×10^{-6}	2.7087×10^{-9}	1.5700×10^{-12}
0.8	1.2677×10^{-5}	1.8660×10^{-7}	7.8960×10^{-9}	2.8500×10^{-12}
0.9	1.5907×10^{-5}	3.7837×10^{-4}	2.7087×10^{-8}	1.0000×10^{-12}
1.0	1.4658×10^{-3}	4.8176×10^{-4}	1.1801×10^{-7}	9.4040×10^{-11}

$y(0) = 0, h = 0.1$

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Problem 2.

$y' = x^2 y^2$

$y(0) = 1, \quad h = 0.1$

$$y(x) = \frac{-3}{x^3 - 3}$$

6. Conclusion

The newly block methods at $k = 3$ and 4 are suitable for nonlinear first order IVPs of the form (1.0). Each of the block Linear Multistep methods display their superiority over existing methods. We observed that each of their equivalent Runge –Kutta type methods performed excellently with the two nonlinear problems tested with these methods see tables 3 and 4.

7. ACKNOWLEDGEMENTS

The author would like to thank the anonymous referee for their constructive criticisms and comments in improving the quality of this manuscript.

8. REFERENCES

[1] Onumanyi P, Awoyemi D.O , Jator S.N and Sirisena U.W, *New Linear Multistep Methods With Continuous Coefficients for First Order IVPs*, Journal of Nigeria Mathematics Society **13** 37-51, 1994.

[2] Chollom J.P , Ndam J.N and Kumleng G.M,*On some properties of the Block linear multi-step methods*,Science World Journal,**2**(3) 11-17 2007.

[3] Odekunle M.R , Adesanya A.O and Sunday J,*A New Block Integrator For the Solution of Initial Value Problems of First-Order Ordinary Differential Equations*,International Journal of Pure and Applied Science and Technology, **11**(2) 92-100, 2012 b.

[4] Badmus A.M and Mshelia D.W, *Uniform Order Zero-Stable K-Step Block Methods For Initial Value Problems Of Ordinary Differential Equations*,**20**(2) 65-74,2012

[5] Badmus A.M and Mishelia D.W,*Some uniform order block methods for the solution of first order ordinary differential equations*, Journal of the Nigerian Association of Mathematical Physics, **19** 149-154 2011.

[6] Badmus A.M , Yahaya Y.A and Subair A.O,*Implicit Stomercowell k-step Hybrid Block Methods For solution of First Order Ordinary Differential Equations (ODEs)*,International Journal of Applied Mathematical Research,**3**(4) 464-472 2014.

[7] Lambert John D,*Computational Methods in Ordinary Differential Equations*,John Wiley and Sons,**New York**, 1973.

[8] Yahaya, Yusuf A,*Some Theories and Applications of Continuous LMM for Ordinary Differential Equations*,PhD Thesis (Unpublished),**University of Jos, Nigeria**, 2004.

[9] Ademola M Badmus and Mohammed S Mohammed,*Construction of Some Implicit Linear Multi- Step Methods into Runge-Kutta Types for Initial Value Problems (IVP) of First Order Differential Equations*,Academy Journal of Science and Engineering,**10**(1) 73-87 2016.

[1]DEPARTMENT OF MATHEMATICAL SCIENCES, NIGERIAN DEFENCE ACADEMY, KADUNA NIGERIA
E-mail address: ambadmus@nda.edu.ng

[2]DEPARTMENT OF MATHEMATICAL SCIENCES, NIGERIAN DEFENCE ACADEMY, KADUNA NIGERIA
E-mail address: sb.jayone@gmail.com