Journal of the Nigerian Mathematical Society
Vol. 35, pp. 139-162, 2017

A CONTINUOUS RUNGE-KUTTA-NYSTRÖM COLLOCATION METHOD WITH TRIGONOMETRIC COEFFICIENTS FOR PERIODIC INITIAL VALUE PROBLEMS

J. O. EHIGIE[1], S. N. JATOR AND S. A. OKUNUGA

ABSTRACT. In this paper, we construct a two-step continuous multistep scheme of Numerov type with two off-grid points via multistep collocation technique. With the generation of several discrete multistep schemes of Numerov type, we show that these discrete methods represent a practical four stage Runge-Kutta-Nyström Collocation Method (RKNCM) with trigonometric coefficients. A detailed analysis of the method such as the stability plots as well as the phase properties of the RKNCM are investigated and presented. Numerical experiments are carried out to illustrate the high effectiveness of the RKNCM compared with some recent methods in the literature.

Keywords and phrases: Periodic initial value problem; Runge-Kutta; Trigonometrical basis, Hybrid, Trigonometric polynomials; Collocation technique.
2010 Mathematical Subject Classification: 65L05, 65L06

1. Introduction

Periodic Initial Value Problems (IVPs) are problems encountered in several areas of engineering and science, such as celestial mechanics, circuit theory, control theory, chemical kinetics, and biology. Many of such equations are the Schrödinger equation, Duffing equations, pleiades problem, orbital problems to mention a few.

In this paper, the direct numerical solution to the second order initial value problem of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \tag{1}$$

Received by the editors June 28, 2016; Revised: February 27, 2017; Accepted: March 01, 2017
www.nigerianmathematicalsociety.org
[1]Corresponding author

is considered. Some researchers have numerically solved (1) by first reducing it to a system of first order differential equations before solving with the Runge-Kutta methods or Linear multistep methods (see Refs [15], [20]). However, this paper presents an approach for the direct solution of (1) without reducing to first order systems. This is because a direct approach by Nyström methods reduces the computational cost by 25 percent. [25].

The classical Runge-Kutta-Nyström methods for solving directly is given by

$$\begin{cases} y_{n+1} = y_n + hy' + h^2 \sum_{j=1}^{s} b_j f(x_n + c_j h, Y_j), \\ y'_{n+1} = y'_n + h \sum_{j=1}^{s} \overline{b} f(x_n + c_j h, Y_j), \\ Y_j = y_n + c_j h y'_n + h^2 \sum_{k=1}^{s} a_{jk} f(x_n + c_k h, Y_k), \quad j = 1, 2, \cdots, s. \end{cases} \tag{2}$$

which can be represented in a Butcher array in the form

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \\ \hline & \overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_s \end{array}$$

The P- stability of a numerical method was a condition discovered by Lambert and Watson [21] to ascertain that a numerical solution has exactly the same behavior with theoretical solution to oscillatory systems. Also, it ensures that numerical solution copes with periodic problems independently of the stepsize used in the implementation. In order to avoid such analysis, the approach introduced by Gautschi [14] for multistep methods and later extended by Lyche [22] seems more attractive because of the trigonometric fitting in the numerical method.

Ozawa [26], derived a class of the functional fitted Runge-Kutta Nyström via order conditions approach [15], while Coleman and Duxbury [6] derived some class of methods via mixed collocation approach including the first derivative in collocation conditions, (see Refs [9], [19]). Alternatively, we derive a Runge-Kutta-Nyström collocation method with trigonometric coefficients via multistep collocation technique from a continuous two-step Numerov scheme, whose coefficients are functions of frequency and the step size such that the exact solutions of the IVPs are obtained if the frequencies can be estimated or are known in advance [20].

Following Jator [19], via multistep collocation technique, we develop a continuous Numerov type formula given by

$$\begin{cases} y(x) = \widetilde{\alpha}(x,v)y_n + \widehat{\alpha}(x,v)y_{n-1} + h^2 \sum_{i=1}^{s} \beta_i(x,v) f_{n+c_i} \\ y'(x) = \frac{d}{dx}\left(\widetilde{\alpha}(x,v)y_n + \widehat{\alpha}(x,v)y_{n-1} + h^2 \sum_{i=1}^{s} \beta_i(x,v) f_{n+c_i}\right) \end{cases} \tag{3}$$

such that evaluation at some specific points yields the two-step multistep methods of Numerov type which are transformed to a trigonometrically fitted Runge-Kutta-Nyström formula by an appropriate transformation. The resulting Runge-Kutta-Nyström Method (RKNM) scheme will be represented in a Butcher array given by

$$\begin{array}{c|cccc} c_1 & a_{11}(v) & a_{12}(v) & \cdots & a_{1s}(v) \\ c_2 & a_{21}(v) & a_{22}(v) & \cdots & a_{2s}(v) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1}(v) & a_{s2}(v) & \cdots & a_{ss}(v) \\ \hline & \bar{b}_1(v) & \bar{b}_2(v) & \cdots & \bar{b}_s(v) \\ \hline & b_1(v) & b_2(v) & \cdots & b_s(v) \end{array}$$

Several methods on trigonometrically fitted Runge-Kutta-Nyström method can be found in literatures [[7], [14], [24], [26]- [30]]. A detailed survey of methods is presented in Paternoster [28] and the references therein. In this paper, the aim is to derive a four-stage Runge-Kutta-Nyström method with First Stage As Last (FSAL) property from a two-step continuous multistep scheme with trigonometric coefficients. To avoid large accumulation of errors, the numerical method shall be implemented in a block by block fashion as implemented in Refs [[1], [8], [9], [18], [19], [24]].

The rest of the paper is organized as follows: The derivation of the continuous multistep method and the corresponding trigonometrically fitted Runge-Kutta-Nyström is presented in section 2. The stability analysis such as the stability plot, P-stability and the order of dispersion are discussed in section 3. The error analysis is shown is section 4. Finally, some numerical experiments are considered to show the numerical performance of the methods in section 5.

2. The continuous method, its block multistep extension and the corresponding Runge-Kutta-Nyström Formula

2.1. Theoretical procedure. Let initial value problem (1) be such that it satisfies all the necessary requirements for the existence of the uniqueness of solution. Now, let y_n be an approximate solution to $y(x)$ at x_n. The numerical solution at point $x_{n+1} = x_n + h$ is of interest where h is the stepsize. Hence, we assume that the numerical solution is represented by the continuous formulation

$$y(x) = \widetilde{\alpha}(x, v)y_n + \widehat{\alpha}(x, v)y_{n-1} + h^2 \sum_{i=1}^{4} \beta_i(x, v) f_{n+c_i} \tag{4}$$

where $\widetilde{\alpha}(x, v)$, $\widehat{\alpha}(x, v)$, $\beta_i(x, v)$, $i = 1, 2, 3, 4$ are the continuous coefficients that must be uniquely determined. We assume that $y_{n+j} \approx y(x_n + jh)$, $y_{n+c_j} \approx y(x_n + c_j h)$, $f_{n+j} \approx y''(x_n + c_j h)$ and $f_{n+c_j} \equiv f\left(x_n + c_j h, y_{n+c_j}\right)$. Eqn (4) facilitated with the derivative of (4) given by

$$z(x, v) = y'(x, v) = \frac{d}{dx}\left(\widetilde{\alpha}(x, v)y_n + \widehat{\alpha}(x, v)y_{n-1} + h^2 \sum_{i=1}^{4} \beta_i(x, v) f_{n+c_i}\right) \tag{5}$$

is evaluated at some points to obtain a block multistep extension in the form

$$AY_m = By_{m-1} + h^2 CF_{m-1} + h^2 DF_m \tag{6}$$

where

$$\begin{aligned}
Y_m &= [y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, z_{n+1}]^T, \\
Y_{m-1} &= [y_{n-\frac{2}{3}}, y_{n-\frac{1}{3}}, y_n, z_n]^T, \\
F_m &= [f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, z_{n+1}]^T. \\
F_{m-1} &= [f_{n-\frac{2}{3}}, f_{n-\frac{1}{3}}, f_n, f'_n]^T,
\end{aligned}$$

and A, B, C, D and E are 4×4 square matrices which are functions of v, (see Ref [12]). The methods (6) obtained from the continuous method (4) and (5) will be transformed to a corresponding Runge-Kutta-Nyström formula.

2.2. Construction of the method. In this section, we present the construction of the continuous Runge-Kutta-Nyström formulation (4) and (5) for arbitrary abscissa $c_i = \{c_1, c_2, c_3, c_4\}$ by the multistep collocation technique described in section 2.1.

We proceed by seeking to approximate the exact solution by the interpolating function of the form

$$y(x) = A\cos\omega(x-x_n) + B\cos\omega(x-x_n) + \sum_{j=0}^{3} a_j (x-x_n)^j \quad (7)$$

$$= \sum_{i=0}^{5} \gamma_j . \varphi_j(x) \quad (8)$$

where

$$\gamma_j = \{A, B, a_0, a_1, a_2, a_3\}$$

and the basis function

$$\varphi_j(x) = \{\cos(\omega(x-x_n)), \sin(\omega(x-x_n)), (x-x_n)^i, i = 0,1,2,3\}$$

.

Let the collocation conditions be imposed such that the interpolating function (7) satisfies the following

$$y(x_{n+j}) = y_{n+j}, \qquad j = 0, -1, \quad (9)$$

$$h^2 y''(x_{n+c_j}) = h^2 f_{n+c_j} \quad j = 1,2,3,4, \quad (10)$$

with $v = \omega h$, (9) and (10) leads to a system of six equations with six unknown parameters to be uniquely determined which takes the matrix form

$$T(v)\gamma_j = \Phi_j, \quad (11)$$

where $T(v)$ is a matrix obtained from the interpolation and collocation of the basis function $\varphi_j(x)$ given by

$$T(v) = h^2 \begin{pmatrix} \frac{\varphi_0(x_n)}{h^2} & \frac{\varphi_1(x_n)}{h^2} & \frac{\varphi_2(x_n)}{h^2} & \frac{\varphi_3(x_n)}{h^2} & \frac{\varphi_4(x_n)}{h^2} & \frac{\varphi_5(x_n)}{h^2} \\ \frac{\varphi_0(x_{n-1})}{h^2} & \frac{\varphi_1(x_{n-1})}{h^2} & \frac{\varphi_2(x_{n-1})}{h^2} & \frac{\varphi_3(x_{n-1})}{h^2} & \frac{\varphi_4(x_{n-1})}{h^2} & \frac{\varphi_5(x_{n-1})}{h^2} \\ \varphi_0''(x_{n+c_1}) & \varphi_1''(x_{n+c_1}) & \varphi_2''(x_{n+c_1}) & \varphi_3''(x_{n+c_1}) & \varphi_4''(x_{n+c_1}) & \varphi_5''(x_{n+c_1}) \\ \varphi_0''(x_{n+c_2}) & \varphi_1''(x_{n+c_2}) & \varphi_2''(x_{n+c_2}) & \varphi_3''(x_{n+c_2}) & \varphi_4''(x_{n+c_2}) & \varphi_5''(x_{n+c_2}) \\ \varphi_0''(x_{n+c_3}) & \varphi_1''(x_{n+c_3}) & \varphi_2''(x_{n+c_3}) & \varphi_3''(x_{n+c_3}) & \varphi_4''(x_{n+c_3}) & \varphi_5''(x_{n+c_3}) \\ \varphi_0''(x_{n+c_4}) & \varphi_1''(x_{n+c_4}) & \varphi_2''(x_{n+c_4}) & \varphi_3''(x_{n+c_4}) & \varphi_4''(x_{n+c_4}) & \varphi_5''(x_{n+c_4}) \end{pmatrix}$$

$$\gamma_j = [A, B, a_0, a_1, a_2, a_3]^T,$$

$$\Phi_j = [y_n, y_{n-1}, h^2 f_{n+c_1}, h^2 f_{n+c_2}, h^2 f_{n+c_3}, h^2 f_{n+c_4}]^T$$

Theorem 2.1. *Given condition* (9) *and* (10), *the continuous formulations* (4) *and* (5) *respectively are equivalent to the following:*

$$y(x) = \sum_{j=0}^{5} \frac{det(T_j(v))}{det(T(v))}\varphi_j(x) \tag{12}$$

and

$$y'(x) = \frac{d}{dx}\left(\sum_{j=0}^{5} \frac{det(T_j(v))}{det(T(v))}\varphi_j(x)\right) \tag{13}$$

where matrix $T_j(v)$ *is the matrix obtained by replacing the* j^{th} *column of* $T(v)$ *by matrix* Γ_j *and*
$\varphi_j(x) = \{\cos(\omega(x-x_n)), \sin(\omega(x-x_n)), 1, (x-x_n), (x-x_n)^2, (x-x_n)^3\}$

Proof:

If we require that the method (4) be defined by the assumed basis functions $\varphi_j(x)$

$$\begin{cases} \widetilde{\alpha}(x,v) = \sum_{i=0}^{5} \widetilde{\alpha}_{i+1,j}(v)\varphi_i(x), & j=0 \\ \widehat{\alpha}(x,v) = \sum_{i=0}^{5} \widehat{a}_{i+1,j}(v)\varphi_i(x), & j=-1 \\ h^2\beta_j(x) = \sum_{i=0}^{5} h^2\beta_{i+1,c_j}(v)\varphi_i(x), & j=1,2,3,4 \end{cases} \tag{14}$$

then the coefficients $\widetilde{\alpha}_{i+1,0}(v)$, $\widehat{\alpha}_{i+1,-1}(v)$ and $\beta_{i+1,c_j}(v)$ are undetermined elements to be determined.

Substituting (14) in (4) we obtain

$$y(x) = \sum_{i=0}^{5} \widetilde{\alpha}_{i+1,0}(v)\varphi_i(x)y_n + \sum_{i=0}^{5} \widehat{\alpha}_{i+1,-1}(v)\varphi_i(x)y_{n-1} + \sum_{j=1}^{4}\sum_{i=0}^{5} h^2\beta_{i+1,c_j}(v)\varphi_i(x)f_{n+c_j} \tag{15}$$

which can equally be expressed as

$$y(x) = \sum_{i=0}^{5} \gamma_i(v)\varphi_i(x) \tag{16}$$

where

$$\gamma_i(v) \equiv \widetilde{\alpha}_{i+1,0}(v)y_n + \widehat{\alpha}_{i+1,-1}(v)y_{n-1} + \sum_{j=1}^{4} h^2\beta_{i+1,c_j}(v)f_{n+c_j}, \quad i=0,1,\ldots,5 \tag{17}$$

Now, if condition (9) and (10) are satisfied on (16), a system of six equation in matrix form

$$T(v)\cdot\gamma_j = \Phi_j \tag{18}$$

where γ_j are to be uniquely determined via Crammers rule. Therefore,

$$\gamma_j = \frac{det(T_j(v))}{det(T(v))} \qquad j = 0, 1, ..., 5$$

where $T_j(v)$ is obtained by replacing the jth column of matrix $T(v)$ by Φ_j. Hence, the continuous multistep scheme is obtained by substitution γ_j in (8). Hence we have

$$y(x) = \sum_{j=0}^{5} \frac{det(T_j(v))}{det(T(v))} \varphi_j(x) \tag{19}$$

and

$$z(x) = y'(x) = \frac{d}{dx} \left(\sum_{j=0}^{5} \frac{det(T_j(v))}{det(T(v))} Q_j(x) \right) \tag{20}$$

Remark 2.2 It is of interest to mention that continuous methods (19) and (20) which are equivalent forms of (4) and (5), respectively for $s = 4$ are used to produce several two-step hybrid multistep methods with trigonometric coefficients which can be expressed in the general linear methods of Butcher.

2.3. **Specification of the method.** In this section, we develop a four-stage RKNCM with four abscissae $c_i = \{c_1, c_2, c_3, c_4\}$, from a continuous two-step hybrid method with trigonometric coefficients. Following (9) and (10), we obtain a system of six equations in matrix form given by

$$T(v)\gamma_j = \Phi_j$$

where

$$T(v) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ \cos(v) & -\sin(v) & 1 & -1 & 1 & -1 \\ -v^2\cos(c_1 v) & -v^2\sin(c_1 v) & 0 & 0 & 2\,h^2 & 6\,h^3 c_1 \\ -v^2\cos(c_2 v) & -v^2\sin(c_2 v) & 0 & 0 & 2\,h^2 & 6\,h^3 c_2 \\ -v^2\cos(c_3 v) & -v^2\sin(c_3 v) & 0 & 0 & 2\,h^2 & 6\,h^3 c_3 \\ -v^2\cos(c_4 v) & -v^2\sin(c_4 v) & 0 & 0 & 2\,h^2 & 6\,h^3 c_4 \end{pmatrix},$$

$$\gamma_j = [A, B, a_0, a_1, a_2, a_3]^T,$$

$$\Phi_j = [y_n, y_{n-1}, h^2 f_{n+c_1}, h^2 f_{n+c_2}, h^2 f_{n+c_3}, h^2 f_{n+c_4}]^T$$

Theorem 2.2. *Let* $0 \leq c_1 < c_2 < \cdots < c_4 \leq 1$, *the matrix function* $T(v)$ *is nonsingular.*

Proof:
We note that if determinant of matrix function $T(v)$ is nonsingular then γ_j can be uniquely determined by elementary algebra. From row reduction, it is easily seen that

$$\det(T(v)) = 12h^5v^4. \begin{vmatrix} cos(c_1v) & sin(c_1v) & 1 & c_1 \\ cos(c_2v) & sin(c_2v) & 1 & c_2 \\ cos(c_3v) & sin(c_3v) & 1 & c_3 \\ cos(c_4v) & sin(c_4v) & 1 & c_4 \end{vmatrix} \tag{21}$$

which is a special case of results obtained in Coleman and Duxbury [6]. Using the technique applied to the Vandermonde determinant

$$\det(A(v)) =$$
$$12h^5v^4 \begin{pmatrix} (-\sin(c_2v - c_3v) + \sin(c_2v - c_4v) - \sin(c_3v - c_4v))\, c_1 \\ + (\sin(c_1v - c_3v) - \sin(c_1v - c_4v) + \sin(c_3v - c_4v))\, c_2 \\ + (\sin(c_1v - c_4v) - \sin(c_2v - c_4v) - \sin(c_1v - c_2v))\, c_3 \\ + (\sin(c_1v - c_3v) + \sin(c_1v - c_2v) + \sin(c_2v - c_3v))\, c_4 \end{pmatrix}.$$

Suppose $\det(A(v)) = 0$, we must have that $c_1 = c_2 = c_3 = c_4$ or $c_1 = c_2 = c_3 = c_4 = 0$ which is a contradiction to our assumption that $0 \le c_1 < c_2 < \cdots < c_4 \le 1$. Hence, for the respective conditions c_i, it follows that the determinant is nonzero. The proof is complete.

For a specific Runge-Kutta-Nyström formula with four abscissae $c_i = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. We shall obtain a block multistep methods which shall in turn is transformed to the corresponding trigonometrically fitted Runge-Kutta-Nyström methods by some algebra. We shall obtain three methods from (19) and two methods from (20) which shall be additional methods to balance the over-determined system.

Hence, it is necessary to evaluate (19) at $x = x_{n+\frac{1}{3}}$, $x = x_{n+\frac{2}{3}}$ and $x = x_{n+1}$ respectively. Thus the discrete form of (19) are obtained on evaluation of $y(x_{n+\frac{1}{3}})$, $y(x_{n+\frac{2}{3}})$ and $y(x_{n+1})$ respectively as

$$\begin{aligned} y_{n+\frac{1}{3}} &= \tfrac{4}{3}y_n - \tfrac{1}{3}y_{n-1} + h^2\left(b_{11}f_n + b_{12}f_{n+\frac{1}{3}} + b_{13}f_{n+\frac{2}{3}} + b_{14}f_{n+1}\right), \\ y_{n+\frac{2}{3}} &= \tfrac{5}{3}y_n - \tfrac{2}{3}y_{n-1} + h^2\left(b_{21}f_n + b_{22}f_{n+\frac{1}{3}} + b_{23}f_{n+\frac{2}{3}} + b_{24}f_{n+1}\right), \\ y_{n+1} &= 2y_n - y_{n-1} + h^2\left(b_{31}f_n + b_{32}f_{n+\frac{1}{3}} + b_{33}f_{n+\frac{2}{3}} + b_{34}f_{n+1}\right), \end{aligned} \tag{22}$$

where

$$\begin{cases} b_{11}(v) = \frac{-9\,\cos\left(\frac{2}{3}\,v\right)-18-8\,v^2-9\,\cos\left(\frac{4}{3}\,v\right)+36\,\cos\left(\frac{1}{3}\,v\right)}{-27\,v^2+27\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = \frac{67}{81} - \frac{581}{14580}\,v^2 + \frac{18101}{16533720}\,v^4 - \frac{15161}{850305600}\,v^6 + \cdots \\ b_{12}(v) = \frac{18\,\cos\left(\frac{2}{3}\,v\right)+9\,\cos(v)+18+16\,\cos\left(\frac{1}{3}\,v\right)v^2+18\,\cos\left(\frac{4}{3}\,v\right)-63\,\cos\left(\frac{1}{3}\,v\right)+5\,v^2}{-27\,v^2+27\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = -\frac{71}{54} + \frac{833}{9720}\,v^2 - \frac{3581}{1574640}\,v^4 + \frac{144071}{3968092800}\,v^6 + \cdots \\ b_{13}(v) = \frac{-10\,\cos\left(\frac{1}{3}\,v\right)v^2-9\,\cos\left(\frac{2}{3}\,v\right)-18\,\cos(v)+18-9\,\cos\left(\frac{4}{3}\,v\right)+18\,\cos\left(\frac{1}{3}\,v\right)-8\,v^2}{-27\,v^2+27\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = \frac{26}{27} - \frac{7}{135}\,v^2 + \frac{43}{34020}\,v^4 - \frac{527}{27556200}\,v^6 + \cdots \\ b_{14}(v) = \frac{9\,\cos\left(\frac{1}{3}\,v\right)-18+5\,v^2+9\,\cos(v)}{-27\,v^2+27\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = -\frac{41}{162} + \frac{35}{5832}\,v^2 - \frac{2797}{33067440}\,v^4 + \frac{1541}{2380855680}\,v^6 + \cdots \end{cases}, \tag{23}$$

$$\begin{cases} b_{21}(v) = \frac{-36\,\cos\left(\frac{2}{3}\,v\right)-18-35\,v^2-36\,\cos\left(\frac{4}{3}\,v\right)+90\,\cos\left(\frac{1}{3}\,v\right)}{-54\,v^2+54\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = \frac{539}{324} - \frac{929}{11664}\,v^2 + \frac{20689}{9447840}\,v^4 - \frac{24257}{680244480}\,v^6 + \cdots \\ b_{22}(v) = \frac{72\,\cos\left(\frac{2}{3}\,v\right)+36\,\cos(v)-36+70\,\cos\left(\frac{1}{3}\,v\right)v^2+72\,\cos\left(\frac{4}{3}\,v\right)-144\,\cos\left(\frac{1}{3}\,v\right)+20\,v^2}{-54\,v^2+54\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = -\frac{137}{54} + \frac{37}{216}\,v^2 - \frac{619}{136080}\,v^4 + \frac{6403}{88179840}\,v^6 + \cdots \\ b_{23}(v) = \frac{-40\,\cos\left(\frac{1}{3}\,v\right)v^2-36\,\cos\left(\frac{2}{3}\,v\right)-72\,\cos(v)+126-36\,\cos\left(\frac{4}{3}\,v\right)+18\,\cos\left(\frac{1}{3}\,v\right)-35\,v^2}{-54\,v^2+54\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = \frac{209}{108} - \frac{403}{3888}\,v^2 + \frac{55733}{22044960}\,v^4 - \frac{60709}{1587237120}\,v^6 + \cdots \\ b_{24}(v) = \frac{36\,\cos\left(\frac{1}{3}\,v\right)-72+20\,v^2+36\,\cos(v)}{-54\,v^2+54\,\cos\left(\frac{1}{3}\,v\right)v^2} \\ \quad = -\frac{41}{81} + \frac{35}{2916}\,v^2 - \frac{2797}{16533720}\,v^4 + \frac{1541}{1190427840}\,v^6 + \cdots \end{cases}, \tag{24}$$

and

$$\begin{cases} b_{31}(v) = \frac{4\cos\left(\frac{1}{3}v\right)-2\cos\left(\frac{2}{3}v\right)-2\cos\left(\frac{4}{3}v\right)-2v^2}{2\cos\left(\frac{1}{3}v\right)v^2-2v^2} \\ \quad = \frac{5}{2} - \frac{43}{360}v^2 + \frac{149}{45360}v^4 - \frac{1123}{20995200}v^6 + \cdots \\ b_{32}(v) = \frac{4\cos\left(\frac{4}{3}v\right)+4\cos\left(\frac{2}{3}v\right)-2-8\cos\left(\frac{1}{3}v\right)+v^2+4\cos\left(\frac{1}{3}v\right)v^2+2\cos(v)}{2\cos\left(\frac{1}{3}v\right)v^2-2v^2} \\ \quad = -\frac{15}{4} + \frac{37}{144}v^2 - \frac{619}{90720}v^4 + \frac{6403}{58786560}v^6 + \cdots \\ b_{33}(v) = \frac{4-2\cos\left(\frac{4}{3}v\right)-2\cos\left(\frac{2}{3}v\right)-2v^2-2\cos\left(\frac{1}{3}v\right)v^2+4\cos\left(\frac{1}{3}v\right)-4\cos(v)}{2\cos\left(\frac{1}{3}v\right)v^2-2v^2} \\ \quad = 3 - \frac{7}{45}v^2 + \frac{43}{11340}v^4 - \frac{527}{9185400}v^6 + \cdots \\ b_{34}(v) = \frac{v^2+2\cos(v)-2}{2\cos\left(\frac{1}{3}v\right)v^2-2v^2} \\ \quad = -\frac{3}{4} + \frac{13}{720}v^2 - \frac{23}{90720}v^4 + \frac{571}{293932800}v^6 + \cdots \end{cases} . \tag{25}$$

To obtain an additional method to make up the block method, we similarly obtain the discrete form of (20) by evaluation of at $x = x_{n+1}$ to get

$$hy'_{n+1} = y'_n + h^2\left(b_{41}f_n + b_{42}f_{n+\frac{1}{3}} + b_{43}f_{n+\frac{2}{3}} + b_{44}f_{n+1}\right) \tag{26}$$

with

$$\begin{cases} b_{41}(v) = \frac{2\sin\left(\frac{2}{3}v\right)-v\cos\left(\frac{1}{3}v\right)+2\sin\left(\frac{1}{3}v\right)-v}{2v\cos\left(\frac{2}{3}v\right)-2v} \\ \quad = \frac{1}{8} + \frac{1}{1440}v^2 + \frac{1}{181440}v^4 + \frac{31}{587865600}v^6 + \cdots \\ b_{42}(v) = \frac{v\cos\left(\frac{2}{3}v\right)+v\cos\left(\frac{1}{3}v\right)-2\sin\left(\frac{2}{3}v\right)-2\sin\left(\frac{1}{3}v\right)}{2v\cos\left(\frac{2}{3}v\right)-2v} \\ \quad = \frac{3}{8} - \frac{1}{1440}v^2 - \frac{1}{181440}v^4 - \frac{31}{587865600}v^6 + \cdots \\ b_{43}(v) = \frac{v\cos\left(\frac{2}{3}v\right)+v\cos\left(\frac{1}{3}v\right)-2\sin\left(\frac{2}{3}v\right)-2\sin\left(\frac{1}{3}v\right)}{2v\cos\left(\frac{2}{3}v\right)-2v} \\ \quad = \frac{3}{8} - \frac{1}{1440}v^2 - \frac{1}{181440}v^4 - \frac{31}{587865600}v^6 + \cdots \\ b_{44}(v) = \frac{2\sin\left(\frac{2}{3}v\right)-v\cos\left(\frac{1}{3}v\right)+2\sin\left(\frac{1}{3}v\right)-v}{2v\cos\left(\frac{2}{3}v\right)-2v} \\ \quad = \frac{1}{8} + \frac{1}{1440}v^2 + \frac{1}{181440}v^4 + \frac{31}{587865600}v^6 + \cdots \end{cases} . \tag{27}$$

It is instructive to note here that the system is over-determined by inclusion of the back value y_{n-1}. Hence, we generate another method, which is added to the initial block to make the system self-starting. So, evaluating (20) at $x = x_n$, that is $y'(x_n)$, we perform some algebra to obtain

$$y_{n-1} = y_n - hy'_n + h^2\left(b_{51}f_n + b_{52}f_{n+\frac{1}{3}} + b_{53}f_{n+\frac{2}{3}} + b_{54}f_{n+1}\right) \tag{28}$$

with

$$\begin{cases} b_{51}(v) = \frac{2\sin\left(\frac{2}{3}v\right)-3\sin\left(\frac{1}{3}v\right)v^2-2\sin\left(\frac{5}{3}v\right)+2v\cos\left(\frac{2}{3}v\right)}{-4\sin\left(\frac{1}{3}v\right)v^2+2\sin\left(\frac{2}{3}v\right)v^2} \\ \quad = \frac{287}{120} - \frac{403}{3360}v^2 + \frac{2977}{907200}v^4 - \frac{346067}{6466521600}v^6 + \cdots \\ b_{52}(v) = \frac{\begin{pmatrix} -4\sin\left(\frac{2}{3}v\right) + 3\sin\left(\frac{2}{3}v\right)v^2 + 2\sin\left(\frac{1}{3}v\right)v^2 + 2\sin\left(\frac{4}{3}v\right) \\ -4v\cos\left(\frac{2}{3}v\right) + 4\sin\left(\frac{5}{3}v\right) - 2\sin\left(\frac{1}{3}v\right) - 2v\cos\left(\frac{1}{3}v\right) \end{pmatrix}}{-4\sin\left(\frac{1}{3}v\right)v^2+2\sin\left(\frac{2}{3}v\right)v^2} \\ \quad = -\frac{81}{20} + \frac{433}{1680}v^2 - \frac{1031}{151200}v^4 + \frac{4349}{39916800}v^6 + \cdots \\ b_{53}(v) = \frac{\begin{pmatrix} 4\sin\left(\frac{1}{3}v\right) - 3\sin\left(\frac{1}{3}v\right)v^2 - 2\sin\left(\frac{2}{3}v\right)v^2 - 2\sin\left(\frac{5}{3}v\right) \\ +4v\cos\left(\frac{1}{3}v\right) - 4\sin\left(\frac{4}{3}v\right) + 2\sin\left(\frac{2}{3}v\right) + 2v\cos\left(\frac{2}{3}v\right) \end{pmatrix}}{-4\sin\left(\frac{1}{3}v\right)v^2+2\sin\left(\frac{2}{3}v\right)v^2} \\ \quad = \frac{117}{40} - \frac{523}{3360}v^2 + \frac{1147}{302400}v^4 - \frac{4945}{86220288}v^6 + \cdots \\ b_{54}(v) = \frac{-2\sin\left(\frac{1}{3}v\right)+2\sin\left(\frac{1}{3}v\right)v^2+2\sin\left(\frac{4}{3}v\right)-2v\cos\left(\frac{1}{3}v\right)}{-4\sin\left(\frac{1}{3}v\right)v^2+2\sin\left(\frac{2}{3}v\right)v^2} \\ \quad = -\frac{23}{30} + \frac{1}{56}v^2 - \frac{29}{113400}v^4 + \frac{443}{230947200}v^6 + \cdots \end{cases} \tag{29}$$

.

Remark 2.1. *We remark here that as $v \to 0$, these coefficients lead to heavy cancellations which may affect numerical results, hence the Taylor series equivalent are used for problems involving very small frequency* [29]. *We also remark that the coefficients of the RKNCM reduce to a corresponding classical two-step block multistep method as $v \to 0$.*

2.4. **Corresponding Runge-Kutta-Nyström Formula.** Substituting (28) in (23), (24), (25) and (27), the corresponding trigonometrically fitted Runge-Kutta-Nyström method takes the form

$$\begin{cases} y_{n+1} = y_n + hy' + h^2 \sum_{j=1}^{4} b_j f(x_n + c_j h, Y_j), \\ y'_{n+1} = y'_n + h \sum_{j=1}^{4} \bar{b}_j f(x_n + c_j h, Y_j), \\ Y_j = y_n + c_j h y'_n + h^2 \sum_{k=1}^{4} a_{jk} f(x_n + c_k h, Y_k), \quad j = 1, 2, \cdots, 4, \end{cases} \tag{30}$$

which is represented in a Butcher's array given as

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & a_{21}(v) & a_{22}(v) & a_{13}(v) & a_{24}(v) \\ \frac{2}{3} & a_{31}(v) & a_{32}(v) & a_{13}(v) & a_{34}(v) \\ 1 & a_{41}(v) & a_{42}(v) & a_{13}(v) & a_{44}(v) \\ \hline & b_1(v) & b_2(v) & b_3(v) & b_s(v) \\ \hline & \bar{b}_1(v) & \bar{b}_2(v) & \bar{b}_3(v) & \bar{b}_s(v) \end{array}$$

with coefficients

$$\begin{cases} a_{21}(v) = \frac{-5\,v^2 \sin\left(\frac{1}{3}v\right) - 54 \sin\left(\frac{1}{3}v\right) + 54 \sin\left(\frac{2}{3}v\right) - 18\,v\cos\left(\frac{2}{3}v\right)}{54 \sin\left(\frac{2}{3}v\right)v^2 - 108\,v^2 \sin\left(\frac{1}{3}v\right)} \\ \quad = \frac{97}{3240} + \frac{107}{816480}v^2 + \frac{629}{661348800}v^4 + \frac{4633}{523788249600}v^6 + \cdots \\ a_{22}(v) = \frac{\begin{pmatrix} 2\,v^2 \sin\left(\frac{1}{3}v\right) - 108 \sin\left(\frac{2}{3}v\right) + 5 \sin\left(\frac{2}{3}v\right)v^2 \\ +18\,v\cos\left(\frac{1}{3}v\right) + 36\,v\cos\left(\frac{2}{3}v\right) + 54 \sin\left(\frac{1}{3}v\right) \end{pmatrix}}{54 \sin\left(\frac{2}{3}v\right)v^2 - 108\,v^2 \sin\left(\frac{1}{3}v\right)} \\ \quad = \frac{19}{540} - \frac{29}{136080}v^2 - \frac{137}{110224800}v^4 - \frac{859}{87298041600}v^6 + \cdots \\ a_{23}(v) = \frac{\begin{pmatrix} -2 \sin\left(\frac{2}{3}v\right)v^2 + 54 \sin\left(\frac{1}{3}v\right) - 5\,v^2 \sin\left(\frac{1}{3}v\right) \\ -36\,v\cos\left(\frac{1}{3}v\right) - 18\,v\cos\left(\frac{2}{3}v\right) + 54 \sin\left(\frac{2}{3}v\right) \end{pmatrix}}{54 \sin\left(\frac{2}{3}v\right)v^2 - 108\,v^2 \sin\left(\frac{1}{3}v\right)} \\ \quad = -\frac{13}{1080} + \frac{1}{30240}v^2 - \frac{1}{2721600}v^4 - \frac{19}{2771366400}v^6 + \cdots \\ a_{24}(v) = \frac{-54 \sin\left(\frac{1}{3}v\right) + 2\,v^2 \sin\left(\frac{1}{3}v\right) + 18\,v\cos\left(\frac{1}{3}v\right)}{54 \sin\left(\frac{2}{3}v\right)v^2 - 108\,v^2 \sin\left(\frac{1}{3}v\right)} \\ \quad = \frac{1}{405} + \frac{1}{20412}v^2 + \frac{109}{165337200}v^4 + \frac{257}{32736765600}v^6 + \cdots \end{cases}, \tag{31}$$

$$\begin{cases} a_{31}(v) = \frac{-8\,v^2\sin\left(\frac{1}{3}\,v\right)+27\,\sin\left(\frac{2}{3}\,v\right)-18\,v\cos\left(\frac{2}{3}\,v\right)}{27\,\sin\left(\frac{2}{3}\,v\right)v^2-54\,v^2\sin\left(\frac{1}{3}\,v\right)} \\ \quad = \frac{28}{405} + \frac{8}{25515}\,v^2 + \frac{22}{10333575}\,v^4 + \frac{19}{1023023925}\,v^6 + \cdots \\ a_{32}(v) = \frac{\left(\begin{array}{c} 2\,v^2\sin\left(\frac{1}{3}\,v\right) - 54\,\sin\left(\frac{2}{3}\,v\right) + 8\,\sin\left(\frac{2}{3}\,v\right)v^2 \\ +18\,v\cos\left(\frac{1}{3}\,v\right) + 36\,v\cos\left(\frac{2}{3}\,v\right) - 54\,\sin\left(\frac{1}{3}\,v\right) \end{array}\right)}{27\,\sin\left(\frac{2}{3}\,v\right)v^2-54\,v^2\sin\left(\frac{1}{3}\,v\right)} \\ \quad = \frac{22}{135} - \frac{1}{1890}\,v^2 - \frac{1}{340200}\,v^4 - \frac{13}{606236400}\,v^6 + \cdots \\ a_{33}(v) = \frac{\left(\begin{array}{c} -2\,\sin\left(\frac{2}{3}\,v\right)v^2 + 108\,\sin\left(\frac{1}{3}\,v\right) - 8\,v^2\sin\left(\frac{1}{3}\,v\right) \\ -36\,v\cos\left(\frac{1}{3}\,v\right) - 18\,v\cos\left(\frac{2}{3}\,v\right) + 27\,\sin\left(\frac{2}{3}\,v\right) \end{array}\right)}{27\,\sin\left(\frac{2}{3}\,v\right)v^2-54\,v^2\sin\left(\frac{1}{3}\,v\right)} \\ \quad = -\frac{2}{135} + \frac{1}{8505}\,v^2 - \frac{1}{1968300}\,v^4 - \frac{1}{77944680}\,v^6 + \cdots \\ a_{34}(v) = \frac{-54\,\sin\left(\frac{1}{3}\,v\right)+2\,v^2\sin\left(\frac{1}{3}\,v\right)+18\,v\cos\left(\frac{1}{3}\,v\right)}{27\,\sin\left(\frac{2}{3}\,v\right)v^2-54\,v^2\sin\left(\frac{1}{3}\,v\right)} \\ \quad = \frac{2}{405} + \frac{1}{10206}\,v^2 + \frac{109}{82668600}\,v^4 + \frac{257}{16368382800}\,v^6 + \cdots \end{cases}, \tag{32}$$

$$\begin{cases} a_{41}(v) = b_1(v) = \frac{-v^2\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{2}{3}\,v\right)-2\,v\cos\left(\frac{2}{3}\,v\right)}{-4\,v^2\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{2}{3}\,v\right)v^2} \\ \quad = \frac{13}{120} + \frac{1}{2016}\,v^2 + \frac{1}{302400}\,v^4 + \frac{61}{2155507200}\,v^6 + \cdots \\ a_{42}(v) = b_2(v) = \frac{-6\,\sin\left(\frac{2}{3}\,v\right)+\sin\left(\frac{2}{3}\,v\right)v^2+2\,v\cos\left(\frac{1}{3}\,v\right)+4\,v\cos\left(\frac{2}{3}\,v\right)-6\,\sin\left(\frac{1}{3}\,v\right)}{-4\,v^2\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{2}{3}\,v\right)v^2} \\ \quad = \frac{3}{10} - \frac{1}{1260}\,v^2 - \frac{1}{226800}\,v^4 - \frac{13}{404157600}\,v^6 + \cdots \\ a_{43}(v) = b_3(v) = \frac{6\,\sin\left(\frac{1}{3}\,v\right)-v^2\sin\left(\frac{1}{3}\,v\right)-4\,v\cos\left(\frac{1}{3}\,v\right)-2\,v\cos\left(\frac{2}{3}\,v\right)+6\,\sin\left(\frac{2}{3}\,v\right)}{-4\,v^2\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{2}{3}\,v\right)v^2} \\ \quad = \frac{3}{40} + \frac{1}{10080}\,v^2 - \frac{1}{907200}\,v^4 - \frac{19}{923788800}\,v^6 + \cdots \\ a_{44}(v) = b_4(v) = \frac{-2\,\sin\left(\frac{1}{3}\,v\right)-2\,\sin\left(\frac{2}{3}\,v\right)+2\,v\cos\left(\frac{1}{3}\,v\right)}{-4\,v^2\sin\left(\frac{1}{3}\,v\right)+2\,\sin\left(\frac{2}{3}\,v\right)v^2} \\ \quad = \frac{1}{60} + \frac{1}{5040}\,v^2 + \frac{1}{453600}\,v^4 + \frac{79}{3233260800}\,v^6 + \cdots \end{cases}, \tag{33}$$

and

$$\begin{cases} \overline{b}_1(v) = \frac{2\,\sin\left(\frac{2}{3}\,v\right) - v\cos\left(\frac{1}{3}\,v\right) + 2\,\sin\left(\frac{1}{3}\,v\right) - v}{2\,v\cos\left(\frac{2}{3}\,v\right) - 2\,v} \\ \quad = \frac{1}{8} + \frac{1}{1440}\,v^2 + \frac{1}{181440}\,v^4 + \frac{31}{587865600}\,v^6 + \cdots \quad , \\ \overline{b}_2(v) = \frac{v\cos\left(\frac{2}{3}\,v\right) + v\cos\left(\frac{1}{3}\,v\right) - 2\,\sin\left(\frac{2}{3}\,v\right) - 2\,\sin\left(\frac{1}{3}\,v\right)}{2\,v\cos\left(\frac{2}{3}\,v\right) - 2\,v} \\ \quad = \frac{3}{8} - \frac{1}{1440}\,v^2 - \frac{1}{181440}\,v^4 - \frac{31}{587865600}\,v^6 + \cdots \quad , \\ \overline{b}_3(v) = \frac{v\cos\left(\frac{2}{3}\,v\right) + v\cos\left(\frac{1}{3}\,v\right) - 2\,\sin\left(\frac{2}{3}\,v\right) - 2\,\sin\left(\frac{1}{3}\,v\right)}{2\,v\cos\left(\frac{2}{3}\,v\right) - 2\,v} \\ \quad = \frac{3}{8} - \frac{1}{1440}\,v^2 - \frac{1}{181440}\,v^4 - \frac{31}{587865600}\,v^6 + \cdots \quad , \\ \overline{b}_4(v) = \frac{2\,\sin\left(\frac{2}{3}\,v\right) - v\cos\left(\frac{1}{3}\,v\right) + 2\,\sin\left(\frac{1}{3}\,v\right) - v}{2\,v\cos\left(\frac{2}{3}\,v\right) - 2\,v} \\ \quad = \frac{1}{8} + \frac{1}{1440}\,v^2 + \frac{1}{181440}\,v^4 + \frac{31}{587865600}\,v^6 + \cdots \quad , \end{cases} , \tag{34}$$

3. Stability Analysis of the Method

3.1. **Interval of Periodicity, P-Stability and Stability Region.** In the literature, Van der Houwen et al. [17], Coleman [5] and Coleman and Ixaru [7], the stability analysis of numerical integrators for periodic solutions have been discussed extensively. In what follows, we apply their stability theory for the numerical integrator (RKNCM) derived in section 2.

A trigonometrically fitted Runge-Kutta-Nyström method (30) may be written in compact form (see Refs [17])

$$\begin{cases} y_{n+1} = y_n + hy' + h^2\mathbf{b}(v)^T\mathbf{f}(\mathbf{e}x_n + \mathbf{c}h, \mathbf{Y}), \\ y'_{n+1} = y'_n + h\overline{\mathbf{b}}(v)^T\mathbf{f}(\mathbf{e}x_n + \mathbf{c}h, \mathbf{Y}), \\ \mathbf{Y} = \mathbf{e}y_n + \mathbf{c}hy'_n + h^2\mathbf{A}(v)\mathbf{f}(\mathbf{e}x_n + \mathbf{c}h, \mathbf{Y}), \end{cases} . \tag{35}$$

The collocation parameters are the elements of the 4-dimensional vector $\mathbf{c}$, the coefficients of the method form the matrix $\mathbf{b}(v)$ and $\overline{\mathbf{b}}(v)$ and the 4×4 matrix $\mathbf{A}(v)$, $\mathbf{e}$ is the 4-dimensional vectors with unit entries.

If we apply (35) to the test equation $y'' = -\lambda^2 y$, and substitute $z = \lambda h$, (35) takes the form

$$\begin{pmatrix} y_{n+1} \\ hz_{n+1} \end{pmatrix} = M(z^2; v) \begin{pmatrix} y_n \\ hz_n \end{pmatrix} \tag{36}$$

and

$$M(z^2; v) = \begin{pmatrix} 1 - z^2\mathbf{b}(v)^T(I + z^2\mathbf{A}(v))^{-1}\mathbf{e} & 1 - z^2\mathbf{b}(v)^T(I + z^2\mathbf{A}(v))^{-1}\mathbf{c} \\ -z^2\overline{\mathbf{b}}(v)^T(I + z^2\mathbf{A}(v))^{-1}\mathbf{e} & 1 - z^2\overline{\mathbf{b}}(v)^T(I + z^2\mathbf{A}(v))^{-1}\mathbf{c} \end{pmatrix}. \tag{37}$$

The characteristic equation of (37) takes the form

$$\xi^2 - 2R_{nm}(z^2, v)\xi + 1 = 0 \tag{38}$$

and has the stability function

$$R_{nm}(z^2; v) = \frac{1 + c_1 z^2 + c_2 z^4 + c_3 z^6}{1 + d_1 z^2 + d_2 z^4 + d_3 z^6} \tag{39}$$

with

$$c_1 = \frac{\begin{pmatrix} -144\, v^4 \sin\left(\frac{2}{3}\, v\right) + 9\, v^4 \sin(v) + 261\, v^4 \sin\left(\frac{1}{3}\, v\right) \\ -972\, \sin(v)\, v^2 - 4860\, v^2 \sin\left(\frac{1}{3}\, v\right) + 3888\, \sin\left(\frac{2}{3}\, v\right) v^2 \end{pmatrix}}{-972\, v^4 \sin(v) - 4860\, v^4 \sin\left(\frac{1}{3}\, v\right) + 3888\, v^4 \sin\left(\frac{2}{3}\, v\right)},$$

$$c_2 = \frac{\begin{pmatrix} -972\, \sin(v) - 36\, v^4 \sin\left(\frac{1}{3}\, v\right) + 144\, v^3 \cos\left(\frac{2}{3}\, v\right) \\ -4\, v^4 \sin(v) - 900\, \sin\left(\frac{2}{3}\, v\right) v^2 - 4860\, \sin\left(\frac{1}{3}\, v\right) + 72\, v^3 \\ +3888\, \sin\left(\frac{2}{3}\, v\right) - 180\, \cos\left(\frac{1}{3}\, v\right) v^3 + 1368\, v^2 \sin\left(\frac{1}{3}\, v\right) \\ +144\, \sin(v)\, v^2 + 18\, v^4 \sin\left(\frac{2}{3}\, v\right) - 36\, v^3 \cos(v) \end{pmatrix}}{-972\, v^4 \sin(v) - 4860\, v^4 \sin\left(\frac{1}{3}\, v\right) + 3888\, v^4 \sin\left(\frac{2}{3}\, v\right)},$$

$$c_3 = \frac{\begin{pmatrix} 180\, v \cos\left(\frac{1}{3}\, v\right) - 846\, \sin\left(\frac{1}{3}\, v\right) + 612\, \sin\left(\frac{2}{3}\, v\right) \\ -126\, \sin(v) + 36\, v^2 \sin\left(\frac{1}{3}\, v\right) + 36\, v \cos(v) + 4\, \sin(v)\, v^2 \\ -18\, \sin\left(\frac{2}{3}\, v\right) v^2 - 72\, v - 4\, v^3 - 144\, v \cos\left(\frac{2}{3}\, v\right) \end{pmatrix}}{-972\, v^4 \sin(v) - 4860\, v^4 \sin\left(\frac{1}{3}\, v\right) + 3888\, v^4 \sin\left(\frac{2}{3}\, v\right)},$$

$$d_1 = \frac{\begin{pmatrix} -7776\,\sin\left(\frac{2}{3}\,v\right)v^2 + 9720\,v^2\sin\left(\frac{1}{3}\,v\right) - 18\,v^4\sin\left(v\right) \\ -522\,v^4\sin\left(\frac{1}{3}\,v\right) + 1944\,\sin\left(v\right)v^2 + 288\,v^4\sin\left(\frac{2}{3}\,v\right) \end{pmatrix}}{-972\,v^4\sin\left(v\right) - 4860\,v^4\sin\left(\frac{1}{3}\,v\right) + 3888\,v^4\sin\left(\frac{2}{3}\,v\right)},$$

$$d_2 = \frac{\begin{pmatrix} -972\,\sin\left(v\right) - 36\,v^4\sin\left(\frac{1}{3}\,v\right) + 144\,v^3\cos\left(\frac{2}{3}\,v\right) \\ +144\,\sin\left(v\right)v^2 - 4\,v^4\sin\left(v\right) - 900\,\sin\left(\frac{2}{3}\,v\right)v^2 \\ +3888\,\sin\left(\frac{2}{3}\,v\right) - 180\,\cos\left(\frac{1}{3}\,v\right)v^3 + 1368\,v^2\sin\left(\frac{1}{3}\,v\right) \\ -36\,v^3\cos\left(v\right) - 4860\,\sin\left(\frac{1}{3}\,v\right) + 72\,v^3 + 18\,v^4\sin\left(\frac{2}{3}\,v\right) \end{pmatrix}}{-972\,v^4\sin\left(v\right) - 4860\,v^4\sin\left(\frac{1}{3}\,v\right) + 3888\,v^4\sin\left(\frac{2}{3}\,v\right)},$$

$$d_3 = \frac{\begin{pmatrix} 180\,v\cos\left(\frac{1}{3}\,v\right) - 846\,\sin\left(\frac{1}{3}\,v\right) + 612\,\sin\left(\frac{2}{3}\,v\right) \\ -18\,\sin\left(\frac{2}{3}\,v\right)v^2 - 126\,\sin\left(v\right) + 36\,v^2\sin\left(\frac{1}{3}\,v\right) \\ +36\,v\cos\left(v\right) + 4\,\sin\left(v\right)v^2 - 72\,v - 4\,v^3 - 144\,v\cos\left(\frac{2}{3}\,v\right) \end{pmatrix}}{-972\,v^4\sin\left(v\right) - 4860\,v^4\sin\left(\frac{1}{3}\,v\right) + 3888\,v^4\sin\left(\frac{2}{3}\,v\right)}.$$

Definition 3.1. [5]. *Given* λ *and* ω *such that* $z = \lambda h$ *and* $v = \omega h$, *the primary interval of periodicity of a method is the largest interval* $(0, \beta^2)$ *such that* $|R_{nm}(z^2; v)| < 1$ *for* $0 < z^2 < \beta^2$; *If* $|R_{nm}(z^2; v)| < 1$ *for all* $z^2 > 0$, *the method is P-stable. If, when* β *is finite,* $|R_{nm}(z^2; v)| < 1$ *for* $\gamma^2 < z^2 < \delta^2$, *where* $\gamma^2 > \beta^2$, *then the interval* (γ^2, δ^2) *is a second interval of periodicity.*

Definition 3.2. [6]. *A region of stability is a region in the* $z - v$ *plane, throughout which* $|R_{nm}(z^2; v)| \leq 1$.

The $z - v$ plot for the Runge-Kutta-Nyström method for $s = 4$ is presented in Figure 1.

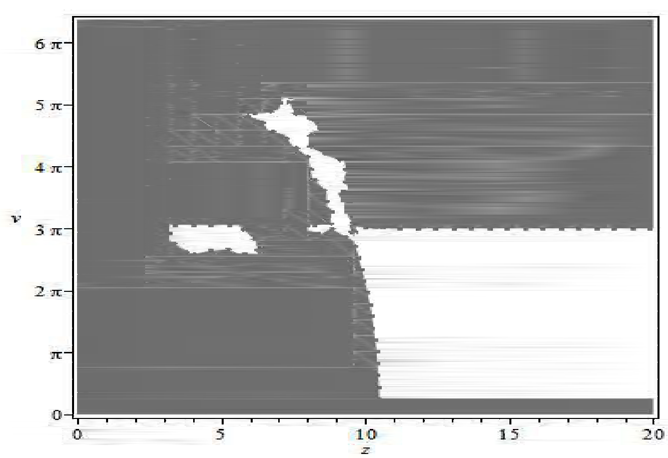

FIGURE 1. $z - v$ plot for the Runge-Kutta-Nyström collocation method

Remark 3.1 (**Remark on *Stability***). *: We remark here that the trigonometrically fitted Runge-Kutta-Nyström method is* not P-stable*; this is because the interval of periodicity* $(0, \beta^2)$ *is finite for* $|R_{nm}(z^2; v)| < 1$*. The primary interval of periodicity for* $|R_{nm}(z^2; v)| < 1$ *as* $v \to 0$ *is* $z \in (0, 3.13^2)$*, while the secondary interval of periodicity is* $(3.13^2, 6.00^2)$.

3.2. **Phase and Amplification Errors.** Ozawa [26] stated that numerical method with trigonometric fitting may be satisfactory for periodic problems even if an inexact value of ω is used. However, it is often required that high accuracies in the phase or amplitude of the solution are sought for. In this section, we shall investigate the phase and amplification errors, when inexact values of ω is used to obtain numerical solution.

For any method having the stability function (39), the quantity $\phi(z^2; v)$ defined by

$$\phi(z^2; v) = z - \cos^{-1}\left(R_{nm}(z^2; v)\right) \tag{40}$$

is called the dispersion (or phase error or phase lag). Modifications from Coleman and Duxbury [6] leads to the following:

Definition 3.3. [6] *If* $\phi(z^2; v) = \gamma(r)z^{q+2} + O(z^{q+4})$ *as* $v \to 0$*, with* $v = rz$ *and* $\gamma(r) \neq 0$*, then the method is said to be dispersive of order* q*.*

In what follows, we investigate this quantity for the Runge-Kutta-Nyström collocation method.

Using R_{nm} in (40) and substituting $v = rz$, we obtain $\phi(z^2; v)$ given by

$$z - \cos^{-1}\left(R_{nm}(z^2; v)\right) = \left(\frac{1}{12960} - \frac{1}{12960}r^2\right) z^6 + O(z^8), \quad v \to 0 \tag{41}$$

Thus we have proved the following theorem:

Theorem 3.1. *The Runge-Kutta-Nyström collocation method with the coefficients evaluated at* ω *is dispersive of order 4.*

Remark 3.2. *For problems, where* $v = z$*, there is no phase lag. In this case,* $\phi(z^2; v) = 0$*.*

4. Local Truncation Error and Order of the Method

We now establish the principal local truncation error of the RKNCM. Let the local truncation errors at x_n be denoted as $\mathcal{L}_\omega^{y_{n+1}}[y(x_n); h]$ and $\mathcal{L}_\omega^{y'_{n+1}}[y(x_n); h]$, when $y(x)$ is the theoretical solution of the problem.

Definition 4.1. [26]. *The order of accuracy of the Runge-Kutta-Nyström method is defined to be* $p = \min\{p_1, p_2\}$ *for the integer* p_1 *and* p_2 *satisfying*

$$\mathcal{L}_{\omega}^{y_{n+1}}[y(x_n); h] = y(x_{n+1}) - y_{n+1} \quad = \quad O(h^{p_1+1}) \tag{42}$$

$$\mathcal{L}_{\omega}^{y'_{n+1}}[y(x_n); h] = y'(x_{n+1}) - y'_{n+1} \quad = \quad O(h^{p_2+1}) \tag{43}$$

where y_{n+1} *and* y'_{n+1} *are the numerical solution given by the initial values* $y_n = y(x_n)$ *and* $y'_n = y'(x_n)$.

Using Definition 4.1, we expand (42) and (43) respectively using the RKNCM (30) with the Taylor series expansion (see Ref. [20]).

$$\begin{aligned}\mathcal{L}_{\omega}^{y_{n+1}}[y(x_n); h] &= y(x_{n+1}) - \left(y_n + hy' + h^2 \sum_{j=1}^{s} b_j f(x_n + c_j h, Y_j) \right) \\ &= -\frac{1}{12960} h^6 \left(y^{(vi)}(x) + \omega^2 y^{(iv)}(x) \right) \end{aligned} \tag{44}$$

$$\begin{aligned}\mathcal{L}_{\omega}^{y'_{n+1}}[y(x_n); h] &= y'(x_{n+1}) - \left(y'_n + h \sum_{j=1}^{s} \bar{b} f(x_n + c_j h, Y_j \right) \\ &= -\frac{1}{6480} h^5 \left(y^{(vi)}(x) + \omega^2 y^{(iv)}(x) \right) \end{aligned} \tag{45}$$

Using the results (44) and (45), we can safely assert that the method is of order 4.

5. Numerical Examples

In this section, the numerical performance of the RKNCM is compared with some numerical methods for periodic initial value problems. We apply it to some linear and nonlinear problems with periodic solutions that have appeared at different times in the literature. The RKNCM is implemented using constant step-size in all the numerical examples. The criteria for comparison are accuracy versus number of function evaluations (NFCN). All computations were carried out by codes using the Maple solvers. We also note that our method is implemented in a block fashion to reduce the number of function evaluations, see Refs. [18].

Problem 5.1: Simos [29]
We consider the nonlinear Duffing equation previously solved in [[4], [6], [19], [29]] for various intervals. The equation is given by

$$y'' + y + y^3 = B \cos \omega x, \qquad y(0) = C_0, \qquad y'(0) = 0. \tag{46}$$

This equation has an analytical solution which takes the form

$$y(x) = C_1 \cos \omega x + C_2 \cos 3\omega x + C_3 \cos 5\omega x + C_4 \cos 7\omega x.$$

where $\omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$, $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-9}$, $C_4 = 0.374 \times 10^{-9}$.

Numerical results of the end-point global errors for the RKNCM is compared with numerical results obtained for some fourth-order methods in [4], [6] for $0 \leq x \leq 40\pi$, while numerical results are compared with the results in Jator et al. [19] for $0 \leq x \leq 300$. We use similar acronyms for results in Jator et al. [19] and Coleman and Duxbury [6] and our numerical results are passed respectively in Tables 1 and 2.

- G2- Fourth-Order mixed collocation method [6]
- E3- Fourth-Order mixed collocation method [6]
- S1- exponentially-fitted Fourth order method [6]
- PE3- Fourth-order polynomial collocation method [6]
- PG2- Fourth-order polynomial collocation method [6]
- C4- Two-step polynomial based hybrid method Chawla et al. [6]
- Chawla- Two-step fourth-order P-stable method [4]
- Simos- Exponentially fitted Runge-Kutta-Nyström method [29]
- Ixaru- Exponentially fitted method [19]

Table 1. Problem 5.1: Maximum errors on [0,40 π]

Method	$h = \frac{\pi}{5}$	$h = \frac{\pi}{10}$	$\frac{\pi}{20}$	$\frac{\pi}{40}$
G2	8.7E-6	4.8E-7	2.9E-8	1.8E-9
E3	1.1E-5	5.8E-7	3.5E-8	2.1E-9
S1	5.4E-5	2.6E-6	1.4E-7	8.7E-9
PE3	1.4E-3	8.7E-5	5.5E-6	3.4E-7
PG2	7.2E-4	4.6E-5	2.9E-6	1.8E-7
C4	3.8E-3	2.0E-4	1.2E-5	7.1E-7
Chawla	4.5E-3	2.9E-4	1.8E-5	1.1E-6
RKNCM	1.2E-6	8.0E-8	5.0E-9	3.2E-10

Remark 5.1. *From Tables 1 and 2, it is easily observed that our methods outperformed the other methods. Numerical results for G2, E3, S1, PE3, PG2, C4 are passed from Coleman and Duxbury* [6].

Problem 5.2: Nonlinear Strehmel-Weiner Problem [23]

TABLE 2. Problem 5.1: Maximum errors on [0,300]

h	TBNM	Simos	Ixaru	RKNCM
1	1.31×10^{-3}	1.70×10^{-3}	1.10×10^{-3}	3.96×10^{-6}
0.5	7.53×10^{-5}	1.88×10^{-4}	4.42×10^{-5}	4.97×10^{-7}
0.25	2.47×10^{-6}	1.37×10^{-5}	1.86×10^{-6}	3.22×10^{-8}
0.125	1.34×10^{-7}	8.70×10^{-7}	6.19×10^{-8}	7.05×10^{-10}
0.0625	8.10×10^{-9}	5.41×10^{-8}	2.40×10^{-9}	1.28×10^{-10}

TABLE 3. Problem 5.2: Numerical results with $\omega = 4$

RKNCM		TBNM [19]		TIRK3 [23]		RADAU5 [15]	
NFCN	Err	NFCN	Err	NFCN	Err	NFCN	Err
600	1.9×10^{-6}	602	2.1×10^{-4}	907	2.5×10^{-4}	853	2.2×10^{-4}
1200	1.2×10^{-7}	1202	1.3×10^{-5}	1288	6.6×10^{-6}	1208	4.4×10^{-4}
1500	5.5×10^{-8}	1602	4.1×10^{-6}	1682	7.0×10^{-6}	1639	6.0×10^{-6}

The nonlinear Strehmel-Weiner problem given by

$$\begin{array}{ll} y'' = (y-z)^3 + 6368y - 6384z + 42\cos 10x & y(0) = 0.5, y'(0) = 0 \\ z'' = -(y-z)^3 + 12768y - 12784z + 42\cos 10x & z(0) = 0.5, z'(0) = 0 \\ & 0 \le x \le 10 \end{array} \tag{47}$$

with an exact solution $y(x) = z(x) = \cos 4x - \frac{\cos 10x}{2}$ is also considered.

We numerically solved (47) with the RKNCM and compared with the Three Stage Trigonometrically Fitted Method [23] (TIRK3), RADAU5 method in Hairer and Wanner [15] and the Trigonometrically Fitted Block Numerov type method (TBNM) [19]. The computational efficiency is measured by the end-point global error with respect to the number of function evaluations (NFCN) used. Details of numerical results are passed in Table 3.

Remark 5.2. *It is seen from Table 3 that the RKNCM yields the most accurate result on implementation in comparison with the methods TBNM* [19]*, TIRK3* [23] *and computations by the RADAU5 code in* [15] *in terms of accuracy. The table further shows that our method is the most efficient because it has the least number of function evaluations with respect to the maximum error obtained.*

Problem 5.3: The wave equation Franco [12]
Lastly, we consider a problem representing a vibrating string with speed

ω given by the partial differential equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - x(1-x)\frac{\partial^2 u}{\partial x^2} + (\omega^2 - 2)u = 0 \qquad 0 < x < 1, \qquad 0 < t \leq 5 \\ u(0,t) = 0, \qquad u(1,t) = 0, \qquad u(x,0) = x(1-x), \qquad u_t(x,0) = 0 \end{cases} \tag{48}$$

where the initial and Dirichlet boundary conditions have been chosen such that the solution is given by $u(x,t) = x(1-x)\cos\omega t$. Equation (48) is transformed by semi-discretization on the spatial variable by using the second-order central difference scheme with parameter $\Delta x = \frac{1}{20}$. the systems of ODE of the form

$$\frac{d^2\mathbf{U}}{dt^2} + k\mathbf{U} = 0 \tag{49}$$

is obtained, where $\mathbf{U}$ denotes the 19-dimensional vector with entries $(u_1, u_2, \cdots, u_{19})$ and K a (positive definite) stiffness matrix with 19 different eigenvalues in the range $[61, 223]$. Numerical computations with $\omega = 5$ for $t_{end} = 5$ have been compared to the methods $M_4\left(\frac{1}{300}, \frac{1}{56}\right)$ given in Franco [12], $M_4\left(\frac{1}{300}, 0\right)$ developed in Chawla and Rao [3], Method $M_4(0,0)$, developed in Chawla [2] and with the highly sophisticated LSODE code for initial value problem as implemented in Hindmarsh [16]. Figure 2 presents the norm error at $t = 5$ in logarithmic scale against computational efficiency of the methods in terms of function evaluations (NFCN).

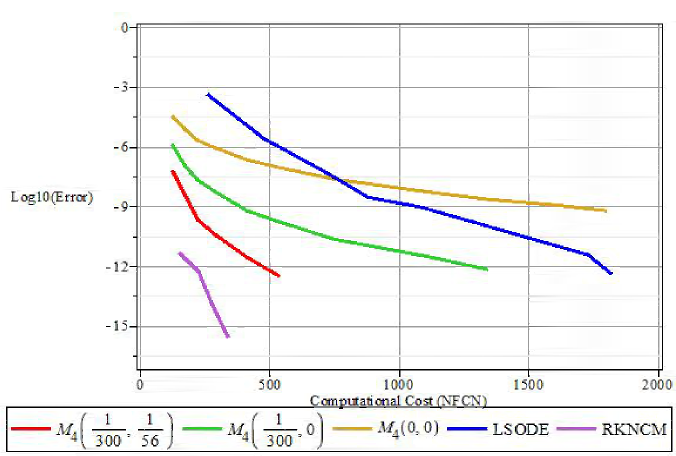

FIGURE 2. *Problem 5.3: Graph of Log10(Error) versus Computational Cost (NFCN)*

Remark 5.3. *It is seen from Figure 2, that new RKNCM is more efficient than the other methods with reduced function evaluation with respect to the accumulated error at $t = 5$.*

6. Conclusion

Another procedure for deriving trigonometrically fitted Runge-Kutta-Nyström method have been introduced using a multistep collocation technique. A practical four-stage RKNCM method was obtained from a class of discrete methods derived from a continuous two-step multistep scheme of Numerov type. Some properties of the methods were investigated. A detailed analysis of the method reveals that the new method is dispersive of order 4 and the stability plot of the method was presented. The implementation of our RKNCM on periodic initial value problems is applied in a block fashion to some linear and nonlinear initial value problems with periodic solutions. Numerical results were presented in terms of accuracy versus number of function evaluations. In the experiments, our new method yields numerical solution with fewer function evaluations, which implies that numerical results are obtained with fewer number of steps for periodic initial value problems, hence it is highly efficient. The method reduces to its classical counterpart when the frequency used in the fitting process is set to zero.

ACKNOWLEDGEMENTS

The authors are particularly grateful to Late Prof. A. B. Sofoluwe for his idea and suggestions. The author would also like to thank the anonymous referee whose comments improved the original version of this manuscript.

References

[1] L. Brugnano, D. Trigiante. *Solving differential problems by multistep initial and boundary value methods*. Gordon and Breach, Amsterdam, 1998.

[2] M. M. Chawla. *Canonical Runge-Kutta-Nyström methods*. Ph.D. Thesis, Universidad de Vallidolid. 1992.

[3] M. M. Chawla and P. S. Rao. A Numerov-type method with minimal phase-lag for the integration of second order periodic initial-value problems. II: explicit methods. *Journal of Computational and Applied Mathematics*, 15 (1986) 329 - 337.

[4] M. M. Chawla and B. Neta. Families of two-step fourth-order P-stable methods for second order differential equations. *Journal of Computational and Applied Mathematics*, 15, (1986), 213 - 223.

[5] J. P. Coleman. Numerical methods for $y'' = f(x, y)$ via rational approximations for the cosine. *IMA Journal of Numerical Analysis*, 9, (1989), 145 - 165.

[6] J. P. Coleman and S.C. Duxbury. Mixed collocation methods for $y'' = f(x, y)$. *Journal of Computational and Applied Mathematics*. 126, (2000) 4775.

[7] J. P. Coleman and L. G. Ixaru. P-stability and exponential fitting methods for $y'' = f(x, y)$. *IMA Journal of Numerical Analysis*, 16, (1996), 179 - 199.

[8] J. O. Ehigie, S. A. Okunuga and A. B. Sofoluwe. 3-point Block Methods for Direct Integration of Second Order Ordinary Differential Equations. *Advances in Numerical Analysis*, Vol. 2011, Article ID 513148, (2011), 14 pages, doi:10.1155/2011/513148.

[9] J. O. Ehigie, S. N. Jator, A. B. Sofoluwe and S. A. Okunuga. Boundary value technique for initial value problems with continuous second derivative multistep method of Enright. *Computational and Applied Mathematics*. 33, (2014), 81 - 93. doi:10.1007/s40314-013-0044-4.

[10] Y. Fang, Y. Song and X. Wu. A robust trigonometrically fitted embedded pair for pertubed oscillators. *Journal of Computational and Applied Mathematics*. 225, (2009), 347 - 355.

[11] S. O. Fatunla. Block methods for second order IVPs. *International Journal of Computer Mathematics*. 41, (1991), 55 -63.

[12] J. M. Franco. An explicit hybrid method of Numerov-type for second-order periodic initial value problems. *Journal of Computational and Applied Mathematics*, 59, (1995), 79 - 90.

[13] J. M. Franco. Runge-Kutta-Nyström methods adapted to the numerical integration of perturbed oscillators. *Computer Physics Communications*. 147, (2002), 770 - 787.

[14] W. Gautschi. Numerical integration of ordinary differential equations based on trigonometric polynomials. *Numerische Mathematik*, 3, (1961), 381 - 397.

[15] Hairer E. and Wanner G., *Solving ordinary differential equations II: stiff and differential algebraic problems* (Second Revised Edition). Germany: Springer Verlag, (1996).

[16] A. C. Hindmarsh. *Odepack, a systematized collection of ODE solvers, in: R. S. Stepleman et al., Eds*, Scientific Computing (North-Holland, Amsterdam, (1983) 55 - 64.

[17] P.J. van der Houwen, B.P. Sommeijer N. Cong, Stability of collocation-based Runge-Kutta-Nyström methods, *BIT* 31 (1991) 469 - 481.

[18] S. N. Jator. Solving Second Order initial value problems by a hybrid multistep methods without predictors. *Applied Mathematics and Computation*, 217, (2010), 4036 - 4046.

[19] S. N. Jator, S. Swindle and R. French. Trigonometrically fitted block Numerov type method for $y'' = f(x, y, y')$. *Numerical Algorithms*, 38(4), (2012), doi: 10.1007/s11075-012-9562-1.

[20] J. D. Lambert. *Computational methods in Ordinary Differential Equations*. John Wiley and sons, New York, (1973).

[21] J. D. Lambert and A. Watson. Symmetric multistep method for periodic initial value problem. *Journal of Institute of Mathematics and its Applications*. 18, (1976) 189202.

[22] T. Lyche. Chebyshevian multistep methods for ordinary differential equations. *Numerische Mathematik*, 19, (1972), 65 - 75.

[23] H. S. Nguyen, R. B. Sidje and N. H. Cong. Analysis of trigonometrically fitted implicit Runge-Kutta methods. *Journal of Computational and Applied Mathematics*, 198, (2007), 187 - 207.

[24] F. F. Ngwane and S. N. Jator. Solving oscillatory problems using a block hybrid trigonometrically fitted method with two off-step point. *Electronic Journal of Differential Equations, Conference*. 20, (2013), 119 -132.

[25] E. J. Nyström, Ueber die numerische Integration von Differentialgleichungen. *Acta Soc. Sci. Fenn.*, 50 (1925) 1-54.

[26] K. Ozawa. A functionally fitted three-stage explicit singly diagonally implicit Runge-Kutta method. *Japan Journal of Industrial and Applied Mathematics*. 22, (2005) 403427.

[27] B. Paternoster. Runge-Kutta (Nyström) methods for ODEs with periodic solutions based on trigonometric polynomials. *Applied Numerical Mathematics*, 28, (1998), 401 - 412.

[28] B. Paternoster, Present state-of-the-art in exponential fitting, *Comput. Phys. Commun.* 183 (2012) 2499–2512.

[29] T. E. Simos. An exponentially-fitted Runge-Kutta method for the numerical integration of initial-value problems with periodic or oscilating solutions. *Computer Physics Communications*, 115, (1998), 1 - 8.

[30] T. E. Simos, E. Dimas and A. B. Sideridis. A Runge-kutta method for the numerical integration of special second-order periodic initial-value problems. *Journal of Computational and Applied Mathematics*, 51, (1994), 317 - 326.

[31] Z. Wang, D. Zhao, Y. Dai and D. Wu. An improved trigonometrically fitted P-stable Obrechkoff method for periodic initial-value problems. *Proceedings of the Royal Society A*, 461, (2005), 1639 - 1658.

COLLEGE OF HORTICULTURE, NANJING AGRICULTURAL UNIVERSITY, NANJING, CHINA AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LAGOS, LAGOS, NIGERIA

E-mail address: jehigie@unilag.edu.ng

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUSTIN PEAY STATE UNIVERSITY, CLARKSVILLE, TN, USA

E-mail addresses: Jators@apsu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LAGOS, LAGOS, NIGERIA

E-mail address: sokunuga@unilag.edu.ng