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ON THE SUBSEMIGROUP GENERATED BY IDEMPOTENTS OF THE SEMIGROUP OF ORDER PRESERVING AND DECREASING CONTRACTION MAPPINGS OF A FINITE CHAIN

M. M. ZUBAIRU

ABSTRACT. Denote $[n]$ to be a finite chain $\{1, 2, \ldots, n\}$ and let $\mathcal{ODP}_n$ be the semigroup of order preserving and order decreasing partial transformations on $[n]$. Let $\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{Dom } \alpha)\ |x\alpha - y\alpha| \leq |x - y|\}$ be the subsemigroup of partial contraction mappings on $[n]$. Now let $\mathcal{ODCP}_n = \mathcal{ODP}_n \cap \mathcal{CP}_n$. Then $\mathcal{ODCP}_n$ is a subsemigroup of $\mathcal{ODP}_n$ In this paper, we identify the subsemigroup generated by the idempotents in the semigroup of order-preserving and order-decreasing partial contractions $\mathcal{ODCP}_n$. In particular, we characterize the idempotents in the semigroup and study factorization in the subsemigroup generated by the idempotents in $\mathcal{ODCP}_n$. We give a necessary and sufficient condition for product of two idempotents to be an idempotent and otherwise.

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1. INTRODUCTION

Denote $[n]$ to be an $n-$ chain $\{1, 2, \ldots, n\}$. A map α which has domain and image both subsets of $[n]$ is said to be a *transformation*. A transformation α whose domain is a subset of $[n]$ (i.e., Dom $\alpha \subseteq [n]$) is said to be *partial*. If the domain of the transformation is the whole $[n]$, then such a transformation is said to be *full* or *total*. The collection of all partial transformations of $[n]$ is known as the *partial transformation semigroup*, which is usually denoted by $\mathcal{P}_n$. A map

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α in $\mathcal{P}_n$ is said to be *order preserving* (resp., *order reversing*) if (for all $x, y \in \text{Dom } \alpha$) $x \leq y$ implies $x\alpha \leq y\alpha$ (resp., $x\alpha \geq y\alpha$); is *order decreasing* if (for all $x \in \text{Dom } \alpha$) $x\alpha \leq x$; an *isometry* (i.e., *distance preserving*) if (for all $x, y \in \text{Dom } \alpha$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in \text{Dom } \alpha$) $|x\alpha - y\alpha| \leq |x - y|$. An element $x \in \text{Dom } \alpha$ is said to be a *fixed* point of α if $x\alpha = x$, and is a *non-fixed point* of α if $x\alpha \neq x$. The set of all fixed points of α is denoted by fix α whereas the set of all non-fixed points of α is denoted by $n(\alpha)$. i.e., fix $\alpha = \{x \in \text{Dom } \alpha : x\alpha = x\}$ and $n(\alpha) = \{x \in \text{Dom } \alpha : x\alpha \neq x\}$. An element a in a semigroup S is said to be an *idempotent* if $a^2 = a$. The set of all idempotent of a semigroup S is usually denoted by $E(S)$. It is well known that an element $\alpha \in \mathcal{P}_n$ is an idempotent if and only if Im $\alpha =$ fix α. In other words, α is an idempotent if and only if $x_i \in A_i$ for $1 \leq i \leq p$, i.e., the *blocks* A_i are *stationary* [14]. An element a in a semigroup S is said to be *periodic* if there exists $n \in \mathbb{N}$ such a^n is an idempotent. Let S be a semigroup with 0 (zero) element. A non zero element $a \in S$ is said to be a *nilpotent* if there exists $m \in \mathbb{N}$ such that $a^m = 0$, and the smallest $m \in \mathbb{N}$ for which $a^m = 0$ is called *nilpotent degree* of a usually denoted by nildeg a. We shall be writing $\emptyset$ to denote the empty map in $\mathcal{P}_n$ (i.e., the zero element of $\mathcal{P}_n$). For standard concept in semigroup theory and transformation semigroups, we refer the reader to Howie [12, 13], Higgins [18] and Mazorchuk [17], respectively.

2. PRELIMINARY

The full transformation semigroup $\mathcal{T}_n$ (where $\mathcal{T}_n$ denote the semigroup of full transformation on $[n]$), is known to be a regular semigroup as in [[18], p.33. Ex.1]. The idempotents in $\mathcal{T}_n$ do not form a subsemigroup for $n \geq 2$ as shown by Howie [15]. However, Vorob'ev [19], showed that the singular elements in $\mathcal{T}_n$ are expressible as product of idempotents in $\mathcal{T}_n$. It is now an interest to researchers to investigate the algebraic properties of the semigroup generated by the idempotents elements. The semigroup generated by idempotents in $\mathcal{T}_n$ was investigated by Howie [15] in 1966. Products of idempotent in certain semigroup of order preserving maps was as well investigated by Howie *et. al* [14]. The semigroup of all order decreasing partial transformation is denoted by $\mathcal{DP}_n$. Algebraic and combinatorial properties of various subsemigroups of $\mathcal{DP}_n$ were investigated by various authors, see for example [1, 2, 3, 4, 5].

Let

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ |x\alpha - y\alpha| \leq |x - y|\}$$

and

$$\mathcal{OCP}_n = \{\alpha \in \mathcal{CP}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ x \leq y \text{ implies } x\alpha \leq y\alpha\}$$

be the subsemigroups of *partial contractions* and of *order preserving partial contractions* of $[n]$, respectively. Let

$$\mathcal{DCP}_n = \{\alpha \in \mathcal{DP}_n : (\text{for all } x, y \in \text{Dom } \alpha) \ |x\alpha - y\alpha| \leq |x - y|\}$$

be the subsemigroup of order decreasing partial contraction maps on $[n]$, and also let

$$\mathcal{ODCP}_n = \mathcal{DCP}_n \cap \mathcal{OCP}_n.$$

Then $\mathcal{ODCP}_n$ is a subsemigroup of $\mathcal{DCP}_n$ and is the subsemigroup of order preserving and order decreasing partial contractions. It is worth noting that certain combinatorial properties of the semigroup $\mathcal{DCP}_n$ was first discussed by Zubairu and Ali [16]. Moreover, it was also shown in [16] that $\mathcal{DCP}_n$ is not regular and its regular elements were characterized. Some of the earlier researches done on semigroups of contraction mappings on chain in algebraic context can be attributed to [6, 7, 5, 8, 11, 9, 10, 16, 20, 21]. Perhaps, it seems nothing has been done so far on the subsemigroup of order preserving and order decreasing partial contractions $\mathcal{ODCP}_n$. In this paper, we study the idempotents in $\mathcal{ODCP}_n$ and describe the semigroup generated by the idempotents of $\mathcal{ODCP}_n$.

Let α be an element of $\mathcal{ODCP}_n$ and let Dom α, Im α, $h(\alpha)$ and fix α denote, the *domain of* α, *image of* α, $|\text{Im } \alpha|$ and $\{x \in \text{Dom } \alpha : x\alpha = x\}$ (i.e., the *set of fixed points of* α), respectively. For $\alpha, \beta \in \mathcal{ODCP}_n$, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all x in Dom $\alpha\beta$. Without ambiguity, we shall be using the notation $\alpha\beta$ to denote the composition of α and β (i.e., $\alpha \circ \beta$).

Next, it is well known that given any transformation α in $\mathcal{P}_n$, the domain of α is partitioned into blocks by the relation ker $\alpha = \{(x, y) \in \text{Dom } \alpha \times \text{Dom } \alpha : x\alpha = y\alpha\}$ and so as in [14], any $\alpha \in \mathcal{ODCP}_n$ can be expressed as

$$\alpha = \begin{pmatrix} A_1 & \dots & A_p \\ x_1 & \dots & x_p \end{pmatrix} \ (1 \leq p \leq n), \tag{1}$$

where $1 \leq x_1 < x_2 < \ldots < x_p \leq n$ and $A_1 < A_2 < \ldots < A_p$. The sets A_i $(1 \leq i \leq p)$ are the equivalence classes under the relation ker α, i.e., $A_i = x_i\alpha^{-1}$ $(1 \leq i \leq p)$.

3. IDEMPOTENTS AND THEIR PRODUCTS

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We first note the following lemmas from [7] and [4] about idempotent elements in $\mathcal{OCP}_n$ and arbitrary elements in $\mathcal{DP}_n$, respectively.

Lemma 3.1 ([7], Lemma 3.3)**.** *Let $\alpha \in E(\mathcal{OCP}_n)$. Then α can be expressed as*

$$\alpha = \begin{pmatrix} A_1 & \ldots & a_{p-1} & A_p \\ a_1 & \ldots & a_{p-1} & a_p \end{pmatrix},$$

where $a_i \in A_i$ $(1 \leq p \leq n)$.

Lemma 3.2. *Let $\alpha, \beta \in \mathcal{DP}_n$. Then* fix $\alpha\beta =$ fix $\alpha \cap$ fix β*.*

We now characterize the idempotents in $\mathcal{ODCP}_n$ in the lemma below:

Lemma 3.3. *Every idempotent ϵ in $\mathcal{OCP}_n$ is expressible as*

$$\epsilon = \begin{pmatrix} a_1 & \ldots & a_{p-1} & A_p \\ a_1 & \ldots & a_{p-1} & a_p \end{pmatrix},$$

where $a_p = \min A_p$.

Proof. Let $\epsilon \in E(\mathcal{ODCP}_n)$. Notice that $E(\mathcal{ODCP}_n) \subseteq E(\mathcal{OCP}_n)$. Thus, $\epsilon \in E(\mathcal{OCP}_n)$ and by Lemma 3.1, ϵ can be expressed as

$$\begin{pmatrix} A_1 & \ldots & a_{p-1} & A_p \\ a_1 & \ldots & a_{p-1} & a_p \end{pmatrix}.$$

Now, it suffices to show that $A_1 = \{a_1\}$ and $a_p = \min A_p$. Suppose by way of contradiction that there exists $a \in A_1$ such that $a \neq a_1$. Thus either $a < a_1$ or $a > a_1$. Now we consider these cases separately.

If $a < a_1$. Then since ϵ is order decreasing we have $a\epsilon \leq a$ and moreover ϵ is order preserving ensures $a_1 < a \leq a_2$. Thus,

$$|a_2 - a| = a_2 - a = a_2\epsilon - a < a_2\epsilon - a\epsilon = |a_2\epsilon - a\epsilon|.$$

This contradicts the fact that ϵ is a contraction.

If $a > a_1$. Then $a_1 < a < a_2$. This ensures

$$|a_2 - a_1| > |a_2 - a|. \tag{2}$$

Notice that a_1 and a_2 are fixed points. Thus

$$|a_2\epsilon - a\epsilon| = |a_2\epsilon - a_1\epsilon| = |a_2 - a_1| > |a_2 - a| \quad \text{(by equation (2))}.$$

This also contradicts the fact that ϵ is a contraction. Therefore $A_1 = \{a_1\}$, as required.

Now to show $a_p = \min A_p$, suppose by way of contradiction that there exists $b \in A_p$ such that $b \leq x$ for all $x \in A_p$. In particular $b \leq a_p$. Therefore

$$\begin{aligned}|b - a_{p-1}| &= b - a_{p-1} = b - a_{p-1}\epsilon < a_p - a_{p-1}\epsilon \\ &= a_p\epsilon - a_{p-1}\epsilon = b\epsilon - a_{p-1}\epsilon = |b\epsilon - a_{p-1}\epsilon|.\end{aligned}$$

This contradicts the fact that ϵ is a contraction and hence the result follows. □

We now have the following lemma:

Lemma 3.4. *Let* $\epsilon \in E(\mathcal{ODCP}_n)$. *If* $a \in \text{Dom } \epsilon$ *such that* $a \notin \text{fix } \epsilon$. *Then* $a\epsilon = \max \text{ fix } \epsilon$.

Proof. Suppose by way of contradiction that there exists $b \in \text{fix } \epsilon$ such that $b > a\epsilon$. Thus $b\epsilon > a\epsilon$. Notice that ϵ is order preserving, as such $b > a$. Also, since ϵ is order decreasing then $a\epsilon \leq a$. Therefore,

$$|b - a| = b - a = b\epsilon - a \leq b\epsilon - a\epsilon = |b\epsilon - a\epsilon|.$$

This contradicts the fact that ϵ is a contraction. Hence the result. □

It is worth noting that products of idempotents in $\mathcal{ODCP}_n$ is not necessarily an idempotent. For the purpose of illustrations, consider

$$\epsilon = \begin{pmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{pmatrix} \text{ and } \rho = \begin{pmatrix} 1 & 2 & \{3,4,5\} \\ 1 & 2 & 3 \end{pmatrix}$$

idempotents in $\mathcal{ODCP}_5$. The product of ϵ and ρ is $\epsilon\rho = \begin{pmatrix} 1 & \{4,5\} \\ 1 & 3 \end{pmatrix}$, which is not an idempotent. Therefore $E(\mathcal{ODCP}_n)$ is not a semigroup. Now what is the description of the semigroup generated by the idempotents in $E(\mathcal{ODCP}_n)$? To answer this question, we begin with the following lemma, which gives a necessary and sufficient conditions for product of two idempotents in $\mathcal{ODCP}_n$ to be an idempotent.

Throughout the remaining content, we shall refer to $\epsilon, \rho \in E(\mathcal{ODCP}_n)$ as

$$\begin{pmatrix} a_1 & \dots & a_{p-1} & A_p \\ a_1 & \dots & a_{p-1} & a_p \end{pmatrix} \text{ and } \begin{pmatrix} b_1 & \dots & b_{r-1} & B_r \\ b_1 & \dots & b_{r-1} & b_r \end{pmatrix} \; (1 \leq p, r \leq n), \tag{3}$$

respectively, unless otherwise specified. We now have the following lemma.

Lemma 3.5. *Let $\epsilon, \rho \in E(\mathcal{ODCP}_n)$ be as expressed in equation 3. Then $\epsilon\rho$ is an idempotent if and only if for any $i \in \{1, 2, \ldots, p\}$:*

(i) if $a_i = b_j$ for some $j \in \{1, \ldots, r-1\}$ then $a_i\rho = a_i$ or;
(ii) if $a_i \in B_r$ then $a_i\rho = \max(\text{fix } \epsilon \cap \text{fix } \rho)$.

Proof. Suppose $\epsilon\rho$ is an idempotent. Let $i \in \{1, \ldots, p\}$ be such that:

(i) $a_i = b_j$ for some $j \in \{1, \ldots, r-1\}$. Thus $a_i \in \text{fix } \epsilon$ and $a_i \in \text{fix } \rho$. Thus $a_i\rho = a_i$;
(ii) $a_i \in B_r$. Notice that $a_i\epsilon\rho = a_i\rho \in \text{Im } \rho = \text{fix } \rho$. Thus either $a_i\epsilon\rho = a_i$ or $a_i\epsilon\rho \neq a_i$.
If $a_i\epsilon\rho = a_i$ then $a_i \in \text{fix } \epsilon\rho = \text{fix } \epsilon \cap \text{fix } \rho$. i.e.,, $a_i \in \text{fix } \rho$. Therefore $a_i\rho = a_i$.
Now if $a_i\epsilon\rho \neq a_i$, i.e., $a_i \notin \text{fix } \epsilon\rho$. Thus by Lemma 3.4 $a_i\epsilon\rho = \max \text{ fix } \epsilon\rho$. i.e., $a_i\epsilon\rho = \max\ (\text{fix } \epsilon \cap \text{fix } \rho)$, as required.

Conversely, suppose for any $i \in \{1, 2, \ldots, p\}$:

(i) if $a_i = b_j$ for some $j \in \{1, \ldots, r-1\}$ then $a_i\rho = a_i$ or;
(ii) if $a_i \in B_r$ then $a_i\rho = \max(\text{fix } \epsilon \cap \text{fix } \rho)$.

Now let $x \in \text{Dom } \epsilon\rho$. Thus $x \in \text{Dom } \epsilon$ and therefore there are three cases to consider, i.e., $x\epsilon = b_j$ for some $j \in \{1, 2, \ldots, r-1\}$, $x\epsilon \in B_r$, or $x\rho \neq b_j$ $(1 \leq j \leq r-1)$ and $x\rho \notin B_r$.

If $x\epsilon = b_j$ for some $j \in \{1, 2, \ldots, r-1\}$. Then $x(\epsilon\rho)^2 = (x\epsilon\rho)\epsilon\rho = (b_j\rho)\epsilon\rho = b_j\epsilon\rho = x\epsilon^2\rho = x\epsilon\rho$.

If $x \in B_r$. Then $x(\epsilon\rho)^2 = (x\epsilon\rho)\epsilon\rho = (b_r\rho)\epsilon\rho = b_r\epsilon\rho = x\epsilon^2\rho = x\epsilon\rho$.

If $x\rho \neq b_j$ $(1 \leq j \leq r-1)$ and $x\rho \notin B_r$. Then $x(\epsilon\rho) = \emptyset = \emptyset^2 = x(\epsilon\rho)^2$. As such in all the cases, $\epsilon\rho$ is an idempotent, as required. □

We now give the following remark.

Remark 3.6. *Let $\epsilon, \rho \in E(\mathcal{ODCP}_n)$ be as expressed in equation (2). Then the product $\epsilon\rho$ is not an idempotent if and only if there exists $i \in \{1, \ldots, p\}$ such that $a_i \in B_r$, $a_i \neq \max(\text{fix } \epsilon \cap \text{fix } \rho)$ and $a_i \notin \text{fix } \rho$.*

We now have the following lemma.

Lemma 3.7. *Let α be a non idempotent element in $\mathcal{ODCP}_n$. Then $\alpha = \begin{pmatrix} d_1 & \ldots & d_{p-1} & D_p \\ d_1 & \ldots & d_{p-1} & x \end{pmatrix}$ with $x \notin D_p$ $(1 \leq p \leq n)$ if and only if $\alpha = \epsilon_1\epsilon_2$ for some $\epsilon_1, \epsilon_2 \in E(\mathcal{ODCP}_n)$ with $\text{fix } \epsilon_1 \cap \text{fix } \epsilon_2 \neq \emptyset$.*

Proof. Suppose $\alpha = \begin{pmatrix} d_1 & \dots & d_{p-1} & D_p \\ d_1 & \dots & d_{p-1} & x \end{pmatrix} \in \mathcal{ODCP}_n$ with $x \notin D_p$. Let $d_p = \min D_p$. Notice that α is order decreasing and $x \notin D_p$, thus $x < \min D_p$, i.e., $x < d_p$. Thus define $\epsilon_1 = \begin{pmatrix} d_1 & \dots & d_{p-1} & D_p \\ d_1 & \dots & d_{p-1} & d_p \end{pmatrix}$ and $\epsilon_2 = \begin{pmatrix} d_1 & \dots & d_{p-1} & \{x, d_p\} \\ d_1 & \dots & d_{p-1} & x \end{pmatrix}$. Then it easily follows from Lemma 3.4 that $\epsilon_1, \epsilon_2 \in E(\mathcal{ODCP}_n)$. Moreover, one can easily verify that $\alpha = \epsilon_1\epsilon_2$.

Conversely, suppose $\alpha = \epsilon_1\epsilon_2$ for some $\epsilon_1, \epsilon_2 \in E(\mathcal{ODCP}_n)$ with fix $\epsilon_1 \cap$ fix $\epsilon_2 \neq \emptyset$. Using Lemma 3.4 we can express ϵ_1 and ϵ_2 as

$$\begin{pmatrix} a_1 & \dots & a_{p-1} & A_p \\ a_1 & \dots & a_{p-1} & a_p \end{pmatrix} \text{ and } \begin{pmatrix} b_1 & \dots & b_{r-1} & B_r \\ b_1 & \dots & b_{r-1} & b_r \end{pmatrix},$$

respectively. Notice that fix $\epsilon_1\cap$fix $\epsilon_2 \neq \emptyset$. Denote $\{c_1, c_2, \dots, c_{m-1}\}$ where $1 \leq m \leq \min\{r, p\}$. Notice that $\alpha = \epsilon_1\epsilon_2$ is not an idempotent. Thus in line with Remark 3.6 there exists $i \in 1, 2, \dots, p$ such that $a_i \in B_r$, $a_i\epsilon_2 \neq a_i$ and $a_i\epsilon_2 \neq \max(\text{fix } \epsilon_1 \cap \text{fix } \epsilon_2)$. Notice also that ϵ_2 is order preserving, i.e., $c_j < a_i$ for all $j \in \{1, \dots, m-1\}$. In particular, $c_{m-1} < a_i$. Now $a_i\epsilon_1\epsilon_2 = a_i\epsilon_2 \neq a_i$, hence $a_i\epsilon_1\epsilon_2 = y$ for some $y \in [n]$. Let $C_m = \{a_i \ (1 \leq i \leq p) : a_i\epsilon_2 \neq a_i\}$. Then $C_m\epsilon_1\epsilon_2 = y$.

It suffices to show that $y \notin C_m$. Now suppose by way of contradiction that $y \in C_m$. Notice that $C_m\epsilon_1\epsilon_2 = y$. Thus $y\epsilon_1\epsilon_2 = y$ i.e., $y \in$ fix $\epsilon_1\epsilon_2$. Thus fix $\epsilon_1\epsilon_2 = \{c_1, \dots, c_{m-1}, y\} = \text{Im } \epsilon_1\epsilon_2$ which means that $\epsilon_1\epsilon_2$ is an idempotent which is a contradiction. Hence $\alpha = \epsilon_1\epsilon_2 = \begin{pmatrix} c_1 & \dots & c_{m-1} & C_m \\ c_1 & \dots & c_{m-1} & y \end{pmatrix}$ and $y \notin C_m$, as required. □

We now have the following characterization which explains when a product of two idempotents gives a nilpotent.

Lemma 3.8. *Let $\epsilon, \rho \in E(\mathcal{ODCP}_n)$ be as expressed in equation* (3)*. Then $\epsilon\rho$ is a nilpotent if and only if* fix $\epsilon \cap$ fix $\rho = \emptyset$.

Proof. Suppose $\epsilon\rho \in E(\mathcal{ODCP}_n)$ is a nilpotent. Suppose by way of contradiction that fix $\epsilon \cap$ fix $\rho \neq \emptyset$. Now $\epsilon\rho$ is a nilpotent implies there exists $r \in \mathbb{N}$ such that $(\epsilon\rho)^r = \emptyset$. Now since fix $\epsilon \cap$ fix $\rho \neq \emptyset$, it means that fix $\epsilon\rho \neq \emptyset$ i.e., Dom $\epsilon\rho \neq \emptyset$. Now let $x \in$ Dom $\epsilon\rho$ and notice that $x(\epsilon\rho)^r = x\epsilon\rho = x$. i.e., $(\epsilon\rho)^r \neq \emptyset$, a contradiction.

Conversely, suppose fix $\epsilon \cap$ fix $\rho = \emptyset$. This means $a_i \neq b_j$ for all $1 \leq i \leq p$ and $1 \leq j \leq p$. In the product $\epsilon\rho$, there are two cases to consider. i.e., either $a_p \in B_r$ or $a_p \notin B_r$.

If $a_p \in B_r$. Then $a_p\epsilon\rho = a_p\rho = b_r$. i.e., $\epsilon\rho = \begin{pmatrix} A_p \\ b_r \end{pmatrix}$. Notice that $\epsilon\rho$ is order decreasing i.e., $b_r \leq y$ for all $y \in A_p$. In particular, $b_r \leq a_p = \min A_p$. Notice also that $a_p \neq b_r$. This ensures $b_r < a_p$ and as such $b_r \notin A_p$. It therefore follows easily that $\epsilon\rho$ is a nilpotent.

Now if $a_p \notin B_r$. Then $\epsilon\rho = \emptyset$ which is obviously a nilpotent. The proof is now complete. □

In the last two paragraph of the proof of the above lemma, we have actually proved the following.

Lemma 3.9. *If $\alpha \in \mathcal{ODCP}_n$ is a nilpotent expressible as a product of two idempotents in $\mathcal{ODCP}_n$. Then $h(\alpha) \leq 1$ and as such nildeg $\alpha = 2$.*

Now let

$$\sigma = \begin{pmatrix} c_1 & \dots & c_{m-1} & C_m \\ c_1 & \dots & c_{m-1} & y \end{pmatrix} \in \mathcal{ODCP}_n \text{ where } y \notin C_m, \ (1 \leq m \leq n). \tag{4}$$

Then we now have the following lemma which explains the product of nilpotent element of height one and σ in $\mathcal{ODCP}_n$.

Lemma 3.10. *Let $\alpha \in \mathcal{ODCP}_n$ be a nilpotent of height one and $\sigma \in \mathcal{ODCP}_n$ be as expressed in equation (4). Then both $\alpha\sigma$ and $\sigma\alpha$ are nilpotents of height less or equal to one.*

Proof. Let $\alpha \in \mathcal{ODCP}_n$ be a nilpotent of height one and $\sigma \in \mathcal{ODCP}_n$ be as expressed in equation (4). Thus α is of the form $\begin{pmatrix} A \\ x \end{pmatrix}$ where $A \subset [n]$ and $x \notin [n] \backslash A$. Now either $x \in \{c_1, \dots, c_{m-1}\}$ or $x \notin \{c_1, \dots, c_{m-1}\}$.

If $x \in \{c_1, \dots, c_{m-1}\}$. Then $\alpha\sigma = \begin{pmatrix} A \\ c_i \end{pmatrix} = \begin{pmatrix} A \\ x \end{pmatrix}$ for some $i \in \{1, \dots, m-1\}$. This is obviously a nilpotent of height 1.

Now if $x \notin \{c_1, \dots, c_{m-1}\}$. Then either $x \in C_m$ or $x \notin \text{Dom } \sigma$. If $x \in C_m$ then $\alpha\sigma = \begin{pmatrix} A \\ y \end{pmatrix}$. Notice that α is order decreasing and $x \notin A$ implies $x \leq g$ for all $g \in A$. Also σ is order decreasing and in particular $x \notin C_m$ ensures that $y < x$ and as such $y \notin A$. This shows that $\alpha\sigma$ is a nilpotent of height less or equal to one. Moreover, if $x \notin \text{Dom } \sigma$. Then it easily follows that $\alpha\sigma = \emptyset$, and therefore $\alpha\sigma$ is a nilpotent of height zero.

Now for the product $\sigma\alpha$, we consider the following cases: If $c_i \in A$ for some $1 \leq i \leq m-1$. Then let $H = \{c_1, \dots, c_{m-1}\} \cap A$. Then

$H\sigma\alpha = x$. i.e., $\sigma\alpha = \begin{pmatrix} H \\ x \end{pmatrix}$. Notice that $x \notin A$ and $H \subseteq A$. Thus, $x \notin H$ and therefore $\sigma\alpha$ is a nilpotent of height less or equal to one as required.

Now if $y \in A$ then $\sigma\alpha = \begin{pmatrix} C_m \\ x \end{pmatrix}$. Notice that $y \notin C_m$ and σ is order decreasing. It means that $y < c$ for all $c \in C_m$. Also notice that α is order decreasing ensures $x = y\alpha \leq y$. i.e., $x \leq y < c$ for all $c \in C_m$. This means $x \notin C_m$. Therefore $\sigma\alpha$ is a nilpotent of height less or equal to one. Now if $c_i \notin A$ and $y \notin A$. Then obviously $\alpha\sigma = \emptyset$ which is a nilpotent of height zero. The proof is now complete. □

Lemma 3.11. *Let $\alpha \in \mathcal{ODCP}_n$ be a nilpotent of height one and $\epsilon \in \mathcal{ODCP}_n$ be as expressed in equation* (3)*. Then both $\alpha\epsilon$ and $\epsilon\alpha$ are nilpotents of height less or equal to one.*

Proof. Let $\alpha \in \mathcal{ODCP}_n$ be a nilpotent of height one and $\epsilon \in E(\mathcal{ODCP}_n)$ be as expressed in equation (3). Thus α is of the form $\begin{pmatrix} A \\ x \end{pmatrix}$ where $A \subset [n]$ and $x \notin [n] \setminus A$. Now either $x = a_i$ for some $1 \leq i \leq p$ or $x \notin \{a_1, \ldots, a_p\}$.

If $x = a_i$ for some $1 \leq i \leq p$. Then $\alpha\epsilon = \alpha$, which is obviously a nilpotent of rank less or equal to one.

Now if $x \notin \{a_1, \ldots, a_p\}$. Then $\alpha\epsilon = \emptyset$, which is also a nilpotent of rank zero.

Now for the product $\epsilon\alpha$. Let $E = \text{Im } \epsilon \cap A$. Then $\epsilon\alpha = \begin{pmatrix} E \\ x \end{pmatrix}$. Now since $E \subseteq A$ and $x \notin A$ then $x \notin E$. Therefore $\epsilon\alpha$ is a nilpotent of height less or equal to one. □

We now prove the following lemma.

Lemma 3.12. *Let $\epsilon \in E(\mathcal{ODCP}_n)$ be as expressed in equation* (3) *and $\sigma \in \mathcal{ODCP}_n$ be as expressed in equation* (4)*. Then*

(i) *If* $\text{fix}\epsilon \cap \text{fix}\sigma = \emptyset$*. Then both $\epsilon\sigma$ and $\sigma\epsilon$ are nilpotents of height less or equal to one.*

(ii) *If* $\text{fix}\epsilon \cap \text{fix}\sigma \neq \emptyset$*. Then $\epsilon\sigma$ and $\sigma\epsilon$ are either idempotents or of the form $\begin{pmatrix} e_1 & e_2 & \ldots & e_{k-1} & E_k \\ e_1 & e_2 & \ldots & e_{k-1} & y \end{pmatrix}$ where $y \notin E_k$, $(1 \leq k \leq n)$.*

Proof. (i) If $\text{fix}\epsilon \cap \text{fix}\sigma = \emptyset$. For the product $\epsilon\sigma$, if $a_i \notin C_m$ for all $1 \leq i \leq p$, then $\epsilon\sigma = \emptyset$ which is obviously an idempotent of height zero.

Now if Im $\epsilon \cap C_m \neq \emptyset$, denote E to be Im $\epsilon \cap C_m$. Then $\epsilon\sigma = \begin{pmatrix} E \\ y \end{pmatrix}$. Notice that $y \notin C_m$ and $E \subseteq C_m$. Thus $y \notin E$ and as such $\epsilon\sigma$ is a nilpotent of height one.

For the product $\sigma\epsilon$, if $c_i \notin A_p$ for all $1 \leq i \leq m-1$ and $y \notin A_p$. Then obviously $\sigma\epsilon = \emptyset$ which is a nilpotent of height zero.

Now if $c_i \in A_p$ for some $1 \leq i \leq m-1$. Then let $H = \{c_1, \ldots, c_{m-1}\} \cap A_p$. Therefore $\sigma\epsilon = \begin{pmatrix} H \\ a_p \end{pmatrix}$. Notice that $\text{fix}\epsilon \cap \text{fix}\sigma = \emptyset$, as such $c_i \neq a_p$ for all $1 \leq i \leq m-1$. Therefore $a_p \notin H$ and as such $\sigma\epsilon$ is a nilpotent of height one.

Thus both $\epsilon\sigma$ and $\sigma\epsilon$ are nilpotents of height less or equal to one.

(ii) If $\text{fix}\epsilon \cap \text{fix}\sigma \neq \emptyset$. For the product $\epsilon\sigma$, let $K = \text{fix}\epsilon \cap \text{fix}\sigma = \{e_1, \ldots, e_t\}$ where $t \leq \min\{p, m\}$.

Thus either $a_i \in C_m$ for some $t < i \leq p$ or $a_i \notin C_m$ for all $t < i \leq p$. If $a_i \in C_m$ for some $t < i \leq p$, we may let $H_t = \{a_i \in C_m : \ t \leq i \leq p\}$. Then

$$\epsilon\sigma = \begin{pmatrix} e_1 & \ldots & e_t & H_t \\ e_1 & \ldots & e_t & y \end{pmatrix}.$$

It is not difficult to see that $y \notin H_t$ and as such $\epsilon\sigma$ is a nilpotent. Now if $a_i \notin C_m$ for all $t < i \leq p$. Then

$$\epsilon\sigma = \begin{pmatrix} e_1 & \ldots & e_t \\ e_1 & \ldots & e_t \end{pmatrix},$$

which is an idempotent.

Now for the product $\sigma\epsilon$, let $K' = \text{fix } \epsilon \cap \text{fix } \sigma = \{e'_1, \ldots, e'_t\}$ where $t \leq \min\{p, m\}$.

Thus either $y = a_i$ for some $t < i \leq p$ or $y \neq a_i$ and $y \notin A_p$ for all $t < i \leq p$.

If $y = a_i$ for some $t < i \leq p$. Then

$$\sigma\epsilon = \begin{pmatrix} e'_1 & \ldots & e'_t & C_m \\ e'_1 & \ldots & e'_t & y \end{pmatrix}.$$

It is clear that $y \notin C_m$ and as such $\sigma\epsilon$ is a nilpotent. Now if $y \neq a_i$ and $y \notin A_p$ for all $t < i \leq p$. Then

$$\sigma\epsilon = \begin{pmatrix} e'_1 & \ldots & e'_t \\ e'_1 & \ldots & e'_t \end{pmatrix},$$

which is an idempotent and the proof is now complete.

□

We now have the following lemma.

Lemma 3.13. *Let* $\sigma \in \mathcal{ODCP}_n$ *be as expressed in equation* (4) *and* $\tau \in \mathcal{ODCP}_n$ *be express as* $\begin{pmatrix} b_1 & \dots & b_{t-1} & B_t \\ b_1 & \dots & b_{t-1} & g \end{pmatrix}$, $g \notin B_t$. *Then*

(i) If $\mathrm{fix}\sigma \cap \mathrm{fix}\tau = \emptyset$. *Then both* $\tau\sigma$ *and* $\sigma\tau$ *are nilpotents of height less or equal to one.*
(ii) If $\mathrm{fix}\tau \cap \mathrm{fix}\sigma \neq \emptyset$. *Then* $\tau\sigma$ *and* $\sigma\tau$ *are either idempotents or of the form* $\begin{pmatrix} e_1 & \dots & e_{k-1} & E_k \\ e_1 & \dots & e_{k-1} & y \end{pmatrix}$ where $y \notin E_k$, $(1 \leq k \leq n)$.

Proof. The proof is similar to the proof of Lemma 3.12. □

Lemma 3.14. *If* $\sigma \in \mathcal{ODCP}_n$ *as expressed in equation* (4). *Then* σ^2 *is an idempotent.*

Proof. Obviously since $y \notin C_m$, $\sigma^2 = \begin{pmatrix} c_1 & \cdots & c_{m-1} \\ c_1 & \dots & m_{m-1} \end{pmatrix}$ which is an idempotent. □

Remark 3.15. *(i) Product of any two nilpotents in* $\mathcal{ODCP}_n$ *of height one is a nilpotent of height less or equal to one;*
(ii) product of two or more idempotents in $\mathcal{ODCP}_n$ *is an idempotent or a nilpotent of height less or equal to one or is an element of the form of equation* (4).

Now let

$$Z = \{\alpha \in \mathcal{ODCP}_n : \ \alpha \text{ is of the form of equation (4)}\}$$

and also let

$$W = \{\alpha \in (\mathcal{ODCP}_n) : \ \alpha \text{ is a nilpotent with } h(\alpha) \leq 1\}.$$

Then we have actually proved the following result.

Theorem 3.16. *The semigroup* $\langle E(\mathcal{ODCP}_n)\rangle = Z \cup W \cup E(\mathcal{ODCP}_n)$.

4. CONCLUDING REMARKS

We have successfully described the semigroup generated by the idempotents in $\mathcal{ODCP}_n$.

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NOMENCLATURE

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Idempotent generated semigroup
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DEPARTMENT OF MATHEMATICAL SCIENCES, BAYERO UNIVERSITY, KANO, NIGERIA

E-mail address: mmzubairu.mth@buk.edu.ng