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A STUDY OF GENERALIZED EILENBERG-MACLANE SPECTRUM THROUGH NEW Ω-SPECTRUM

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ABSTRACT. In this paper we construct the Eilenberg-MacLane spectrum using Moore space then we define a new Ω spectrum, finally we study the generalization of Eilenberg-MacLane spectrum through new Ω spectrum.

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1. INTRODUCTION

In [4] Dold-Thom showed that for any CW-complex X , $\pi_i(SP^\infty(X)) \cong \tilde{H}_i(X)$. In [7], E.H.Spanier showed that if X is a connected CW-complex then there is a weak homotopy equivalence

$$\rho : SP^\infty(X) \to \Omega SP^\infty(\Sigma X)$$

defined by $\rho(x_1, x_2, ..., x_n, ...)(t) = ((x_1, t), (x_2, t), ..., (x_n, t), ...)$, where ρ is continuous because its restriction to $SP^n(X)$ is continuous for every $n \geq 0$.
In [5], Eckmonn-Hilton showed that for pointed CW-complexes X and Y, $[\Sigma X, Y] \cong [X, \Omega Y]$ where $[X, Y]$ denotes the set of all homotopy class of maps from X to Y.
The functor SP^∞ has the interesting properties that it can be used to define Eilenberg-Maclane Spectrum .

Now we recall the following definitions and statements:-

Definition 1.1:[2] A Hausdorff space X is a CW-complex, if it

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satisfies the following conditions:
i) X is a disjoint union of sets e_n,called cells
ii) To each cell e_n and its closure $\tilde{e_n}$, there exists a continuous function $f : E_n \to \tilde{e_n}$ such that $f|(E_n - S_{n-1})$, where E_n Is a closed n-ball and S_{n-1} its boundary.
iii) $f(S_{n-1}) \subset X^{n-1}$, where $X^{n-1}$is the n-skeleton of X.
iv) For any cell $e_n \subset X$, $e_n \cap \tilde{e} \neq 0$,for at most finitely many cells $e \subset X$.
v) A subset $A \subset X$ is closed iff $A \cap \tilde{e}$ is closed for all cells $e \subset X$.
A pointed CW-complex is called an Eilenberg MacLane space if it has only one nontrivial homotopy group. If G is a group and n is a positive integer, the Eilenberg-MacLane space of type (G,n) is a pointed CW-complex X whose homotopy groups vanish in all dimensions except n , where G = $\pi_n(X)$ and G is to be abelian for n $>$ 1,
we can write the notation K(G,n) for a CW-complex which represents an Eilenberg-MacLane space of type (G,n).

Definiton 1.2:[2, 6] Let X be a topological space with base point $x_0 \in X$.For $n \geq 0$, we define the n fold symmetric product of X , denoted by SP^nX by $SP^0X = x_0, SP^nX = X^n/S_n$ for $n \geq 1$,where X^n denotes the n fold cartesian product of X with itself and S_n denotes the symmetric group on n objects regarding as acting on X^n by permuting the coordinates.
Hence for $n \geq 1$, $SP^nX = \{(x_1, \cdots, x_n) : x_i \in X)\}$,
We define $\lim_{n\to\infty} SP^nX = \cup_{n=1}^{\infty} SP^nX$ is called an infinite symmetric product of X and is denoted by $SP^\infty X$.

P.K.Rana in [1] showed that SP^n and SP^∞ are covariant functor from the category of pointed topological spaces and base point preserving continuous maps to the category of pointed topological spaces and base point preserving continuous maps.

Definition 1.3:[2] Given an abelian group G and $n \geq 1$ if $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for $i \neq n$. then the space X is called a Moore space of type (G, n) and is denoted by $M(G, n)$.

Definition 1.4:[2] Let X be a pointed topological space with base point x_0 .Then the Loop space of X denoted by ΩX, defined to be the space of all continuous pointed map $\alpha : S^1 \to (X, x_0)$ equipped with compact open topology.

Definition 1.5: The spectrum $\underline{X} = \{X_n, \alpha_n\}$ given by $X_n = K(\mathbb{Z}, n)$ and $\alpha_n : K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1)$, a base point preserving weak homotopy equivalence, is called an **Eilenberg-MacLane spectrum**.

Theorem 1.6:[4] For any CW-complex X , $\pi_i(SP^\infty(X)) \cong \tilde{H}_i(X)$.

Proof. Let C be the category of base pointed finite CW-complexes and SP^∞ be the infinite symmetric product functor.Using Dold-Thom Theorem in [4], $\pi_i(SP^\infty(X)) \cong \tilde{H}_i(X)$ on C. □

Theorem 1.7:(Eckmonn-Hilton)[5] For pointed CW-complexes X and Y, $[\Sigma X, Y] \cong [X, \Omega Y]$ where $[X, Y]$ denotes the set of all homotopy class of maps from X to Y.

Proof. Let $\Sigma : \mathbf{C} \to \mathbf{C}$ be the suspension functor and Ω be the loop functor, for any pointed CW-Complexes X and Y ,from [5], it follows. □

2. CONSTRUCTION OF THE SPECTRUM

In this section we construct the Eilenberg-Maclane spectrum using Moore space.To do this we use the following results:

Lemma 2.1:[2] For any CW-complex X , $H_k(X) = \tilde{H}_k(X)$ if $k \neq 0$ and $H_0(X) = \tilde{H}_0(X) \bigoplus Z$

Lemma 2.2:(E.H.Spanier)[6, 7] If X is a connected CW-complex then there is a weak homotopy equivalence

$$\rho : SP^\infty(X) \to \Omega SP^\infty(\Sigma X)$$

defined by $\rho(x_1, x_2, ..., x_n, ...)(t) = ((x_1, t), (x_2, t), ..., (x_n, t), ...)$.

Proof. Since $\rho(x_1, x_2, ..., x_n, ...)(t) = ((x_1, t), (x_2, t), ..., (x_n, t), ...)$, ρ is continuous because its restriction to $SP^n(X)$ is continuous for every $n \geq 0$ and hence ρ is a homomorphism which is one-one mapping. □

Lemma 2.3:[2] Given any abelian group G and an integer $n > 1$ there exists a CW-complex X such that $X = M(G, n)$.

Theorem 2.4: X is a Moore space of type (G, n) if and only if $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$.

Proof. Let X is $M(G, n)$ for $n \geq 1$.Then $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for $i \neq n$. As $n \geq 1$, so by lemma 2.1, $\tilde{H}_i(X) \cong G$ if $i = n$ and $\tilde{H}_i(X) = 0$ if $i \neq n$. Now by Dold-Thom Theorem [4] , we have $\tilde{H}_i(X) \cong \pi_i(SP^{\infty}(X))$.So $\pi_i(SP^{\infty}(X)) \cong G$, if $i = n$ and 0 , if $i \neq n$.Thus $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$.
Conversely , let $SP^{\infty}(X)$ is $K(G, n)$ for $n \geq 1$.Now $\tilde{H}_i(X) \cong \pi_i(SP^{\infty}(X)) \cong G$,if $i = n$ and 0 , if $i \neq n$.Hence $\tilde{H}_i(X) \cong G$, if $i = n$ and 0 , if $i \neq n$. Hence $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for $i \neq n$ as $n \geq 1$. Hence X is $M(G, n)$. □

Now G is any abelian group and $n > 1$ be an integer , then by Lemma 2.3 , there is a CW-complex X such that X is a $M(G, n)$. Now by Theorem 2.4, $SP^{\infty}(X) \cong SP^{\infty}(M(G, n)) = K(G, n)$.

Theorem 2.5: For any given abelian group G and $n > 1$, $SP^{\infty}(\Sigma^k M(G, n))$ is $K(G, n + k)$.

Proof. For $n > 1$, $M(G, n)$ is simply connected and hence connected. Now by Lemma 2.2, there is a weak homotopy equivalence $\rho : SP^{\infty}(M(G, n)) \to \Omega SP^{\infty}(\Sigma M(G, n))$. Using the composite map we can get a weak homotopy equivalence $\rho^k : SP^{\infty}(M(G, n)) \to \Omega^k SP^{\infty}(\Sigma^k M(G, n))$.
Thus $SP^{\infty}(\Sigma^k M(G, n))$ is $K(G, n + k)$. □

Theorem 2.6: Let G be an abelian group and M(G,n) be a Moore space.Then there is a weak homotopy equivalence
$\beta_n : SP^{\infty}(M(G, n)) \to \Omega SP^{\infty}(M(G, n + 1))$

Proof. Since $\pi_{n+1}(SP^{\infty}(M(G, n+1))) \cong \pi_n(\Omega SP^{\infty}(M(G, n+1))) \cong G$.
Again $\pi_n(SP^{\infty}(M(G, n)) \cong \pi_{n+1}(SP^{\infty}(M(G, n + 1)))$, and hence $\pi_n(SP^{\infty}(M(G, n)) \cong \pi_n(\Omega SP^{\infty}(M(G, n + 1))$ for each $n \geq 1$. Hence there is a continuous function
$\beta_n : SP^{\infty}(M(G, n)) \to \Omega SP^{\infty}(M(G, n + 1))$ whose induced homomorphism $\beta_n^* : \pi_n(SP^{\infty}(M(G, n)) \to \pi_n(\Omega SP^{\infty}(M(G, n + 1)))$ is an isomorphism .
Since $\Omega SP^{\infty}(M(G, n+1))$has the homotopy type of a CW-complex and hence β_n is a weak homotopy equivalence. □

Proposition 2.7: If $A_n = SP^\infty(M(G,n))$ then $\{A_n, \beta_n\}$ define an Eilenberg-Maclane Spectrum.

Proof. Since $SP^\infty(M(G,n)) = K(G,n)$, from Definition 1.5 and Theorem 2.6, it follows.
We know every loop space is a H-group and the set of homotopy classes of maps from any pointed space to an H-group admits a group structure and so there is a natural group structure on [X,K(G,n)]. ☐

Now we have the following Theorem:
Theorem 2.8: For all path connected n-dimensional CW-complex X , $[X, M(G,n)] \cong [X, SP^\infty(M(G,n))], n > 1$.

Proof. Since $M(G,n) \subset SP^\infty(M(G,n))$ and so
$i_* : [X, M(G,n)] \to [X, SP^\infty(M(G,n))]$ is injective as $M(G,n)$ is simply connected. Surjectivity follows from the fact that X is n-dimensional CW-complex and cellular approximation. ☐

Corollary 2.9: Given an abelian group G an integer $n \geq 1$ and a path-connected n-dimensional CW-complex X ,then
$H^n(X,G) = [X, M(G,n)] = [X, SP^\infty(M(G,n))]$, is the cohomology group

Proof. Using Theorem 2.4, Theorem 2.8 and corollary of the Hopf theorem in [2], it follows. ☐

3. GENERALIZATION OF THE SPECTRUM

Now to generalized the Eilenberg-Maclane spectrum we can rewrite the spectrum given in Proposition 2.7 as follows
$A_k = \Omega SP^\infty(\Sigma^{k-1}M(G,2))$ and $\alpha_k = \Omega.\beta_{k+1}$, $k > 0$
where β_k is given in Theorem 2.6
Let X be a connected CW-complex ,
define a new spectrum $\boldsymbol{A}$=$\{A_n, \alpha_n\}$, where

$$A_k = \begin{cases} \Omega^{-(k-2)}SP^\infty(X), & k \leq 0 \\ \Omega SP^\infty(\Sigma^{k-1}X), & k > 0 \end{cases} \quad \text{and } \alpha_k = \begin{cases} id, & if\ k \leq 0 \\ \Omega\rho_k, & if\ k > 0 \end{cases}$$

where $\rho_k : SP^\infty(\Sigma^{k-1}X) \to \Omega SP^\infty(\Sigma^k X)$ is the weak homotopy equivalence given in Lemma 2.2.

Cohomology theory associated with the spectrum $\boldsymbol{A}$ is given by
$h^n(Y, \boldsymbol{A}) = [Y, A_n]$
we can find corresponding coefficient system by assuming $Y = S^0$

If $n \leq 0$,
$h^n(S^0, \boldsymbol{A})$
$= [S^0, \Omega^{-(n-2)}SP^\infty(X)]$
$= [\Sigma^{-(n-2)}S^0, SP^\infty(X)]$
$= [S^{-(n-2)}, SP^\infty(X)]$
$= \pi_{-(n-2)}(SP^\infty(X))$
$= H_{-(n-2)}(X, \mathbf{Z})$

$n > 0$,
$h^n(S^0, \boldsymbol{A})$
$= [S^0, \Omega SP^\infty(\Sigma^{n-1}X)]$
$= [\Sigma S^0, SP^\infty(\Sigma^{n-1}X)]$
$= [S^1, SP^\infty(\Sigma^{n-1}X)]$
$= \pi_1(SP^\infty(\Sigma^{n-1}X))$
$= H_1(\Sigma^{n-1}X, \mathbf{Z})$

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CONFLICT OF INTEREST STATEMENT

This paper doesn't have any conflict of interest statement as it is produced and processed under purely private interest.

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