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BIPOLAR PICTURE FUZZY SUBGROUP OF A GROUP

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ABSTRACT. In this paper, the bipolar picture fuzzy set is extended to groups by introducing the concept of bipolar picture fuzzy subgroup of a group. This is a generalisation of both bipolar fuzzy subgroup and bipolar intuitionistic fuzzy subgroup. Several properties of bipolar picture fuzzy subgroup were established. Finally, the notions of bipolar picture fuzzy cosets and bipolar pseudo picture fuzzy cosets were introduced. .

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1. INTRODUCTION

Zadeh [15] pioneered the theory of fuzzy sets (FSs), and since then there are several generalisations and extensions such as intuitionistic fuzzy sets (IFSs) introduced by Atanassov [2, 3], picture fuzzy set (PFS) initiated by Cuong and Kreinovich [6] which has been extensively studied and applied (see [4, 5, 7, 14] for details). Rosenfeld [12] introduced the notion of fuzzy (sub)groups (FGs). Dogra and Pal [8] put forward the concept of picture fuzzy subgroup (PFSG) of a crisp group and obtained some of its properties.

Zhang [16, 17] initiated bipolar fuzzy sets (BFSs) as an extension of Zadeh's work. Lee [9] extended Zhang's work by obtaining some basic operations on BFSs, Anitha et al [1] introduced bipolar fuzzy subgroups of a group, Tahir and Munir [10] studied the bipolar fuzzy group and some basic results were developed by them.

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Muhammad et al [11] introduced the concept of bipolar picture fuzzy set (BPFS) as a combination of BFS and PFS. They established some basic properties and their fundamental operations. They put forward some applications of proposed model to relate it with real life problems such as in Business and Medication.

In this paper, we employ the notion of bipolar picture fuzzy sets to introduce the concept of bipolar picture fuzzy subgroup of a group. This is seen as a generalisation of both bipolar fuzzy subgroup and bipolar intuitionistic fuzzy subgroup of a group. Several properties of bipolar picture fuzzy subgroup were established. Finally, the notion of bipolar picture fuzzy cosets and their basic properties were obtained.

2. PRELIMINARY

This section contains some basic definitions and results.

Definition 1: [15] Let Y be a nonempty set. A FS X of Y is

$$X = \{\langle y, \sigma_X(y)\rangle | y \in Y\} \tag{1}$$

with a membership function

$$\sigma_X : Y \longrightarrow [0,1], \tag{2}$$

where the function $\sigma_X(y)$ is the membership value of $y \in X$.

Definition 2: [6] A PFS J of Y is defined as

$$J = \{(y, \sigma_J(y), \nu_J(y), \gamma_J(y)) | y \in Y\}, \tag{3}$$

where the functions

$$\sigma_J : Y \to [0,1],\ \nu_J : Y \to [0,1] \text{ and } \gamma_J : Y \to [0,1] \tag{4}$$

are called the positive, neutral and negative membership degrees of $y \in J$, respectively, and $\sigma_J, \nu_J, \gamma_J$ satisfy

$$0 \leq \sigma_J(y) + \nu_J(y) + \gamma_J(y) \leq 1,\ \forall y \in Y. \tag{5}$$

For each

$$y \in Y, \mathbb{S}_J(y) = 1 - (\sigma_J(y) + \nu_J(y) + \gamma_J(y)) \tag{6}$$

is called the refusal membership degree of $y \in J$.

Definition 3: [12] Let $(G, \bullet)$ be a group and $X = \{(y, \sigma_X(y)) \mid y \in G\}$ be a FS in G. Then, X is called *fuzzy subgroup* (FSG) of G if

$$\sigma_X(a \bullet b) \geq \sigma_X(a) \wedge \sigma_X(b) \tag{7}$$

and

$$\sigma_X(a^{-1}) \geq \sigma_X(a)\ \forall a, b \in G \tag{8}$$

.

Definition 4: [8] Let $(G, \bullet)$ be a crisp group and J be a PFS in G. Then, J is called picture fuzzy subgroup of G (PFSG) if

(i)

$$\sigma_J(a \bullet b) \geq \sigma_J(a) \wedge \sigma_J(b), \tag{9a}$$

$$\nu_J(a \bullet b) \geq \nu_J(a) \wedge \nu_J(b), \tag{9b}$$

$$\gamma_J(a \bullet b) \leq \gamma_J(a) \vee \gamma_J(b). \tag{9c}$$

(ii)

$$\sigma_J(a^{-1}) \geq \sigma_J(a), \tag{10a}$$

$$\nu_J(a^{-1}) \geq \nu_J(a), \tag{10b}$$

$$\gamma_J(a^{-1}) \leq \gamma_J(a) \ \forall a, b \in G. \tag{10c}$$

Remark 1: Notice that a^{-1} is the inverse of $a \in G$, or equivalently, J is a PFSG of G if and only if

$$\sigma_J(a \bullet b^{-1}) \geq \sigma_J(a) \wedge \sigma_J(b), \tag{11a}$$

$$\nu_J(a \bullet b^{-1}) \geq \nu_J(a) \wedge \nu_J(b), \tag{11b}$$

$$\gamma_J(a \bullet b^{-1}) \leq \gamma_J(a) \vee \gamma_J(b). \tag{11c}$$

Definition 5: [17] Let Y be a nonempty set. A bipolar fuzzy set $\mathbb{B}$ in Y (BFS) is defined as

$$\mathbb{B} = \{\langle y, \sigma^+_{\mathbb{B}}(y), \sigma^-_{\mathbb{B}}(y)\rangle | y \in Y\} \tag{12}$$

where

$$\sigma^+_{\mathbb{B}} : Y \longrightarrow [0,1] \text{ and } \sigma^-_{\mathbb{B}} : Y \longrightarrow [-1,0]. \tag{13}$$

The membership function $\sigma^+_{\mathbb{B}}(y)$ demonstrates the satisfaction degree of an element y to the attribute in favour of a bipolar fuzzy set $\mathbb{B}$ and the membership function $\sigma^-_{\mathbb{B}}(y)$ denotes the degree to which an element y favours some implicit counter-property corresponding of $\mathbb{B}$.

Definition 6: [11] Let Y be a nonempty set. A bipolar picture fuzzy set (BPFS) B in Y is an object of the form

$$B = \{\langle y, \sigma^{\pm}_B(y), \nu^{\pm}_B(y), \gamma^{\pm}_B(y)\rangle | y \in Y\} \tag{14}$$

where

$$\sigma^+_B, \ \nu^+_B, \ \gamma^+_B : B \longrightarrow [0,1] \tag{15}$$

and

$$\sigma^-_B, \ \nu^-_B, \ \gamma^-_B : B \longrightarrow [-1,0] \tag{16}$$

are called positive degree of membership, neutral degree of membership, positive degree of non membership and negative non membership degree, neutral non membership degree, negative non membership degree, respectively with conditions

$$0 \leq \sigma_B^+ + \nu_B^+ + \gamma_B^+ \leq 1 \tag{17}$$

$$-1 \leq \sigma_B^- + \nu_B^- + \gamma_B^- \leq 0 \tag{18}$$

$$0 \leq \sigma_B^+ + \nu_B^+ + \gamma_B^+ - \sigma_B^- - \nu_B^- - \gamma_B^- \leq 2. \tag{19}$$

Definition 7: [11] For all $y \in Y$, let

$$B = \{\langle y, \sigma_B^{\pm}(y), \nu_B^{\pm}(y), \gamma_B^{\pm}(y)\rangle | y \in Y\} \tag{20}$$

and

$$R = \{\langle y, \sigma_R^{\pm}(y), \nu_R^{\pm}(y), \gamma_R^{\pm}(y)\rangle | y \in Y\} \tag{21}$$

be two BPFSs. Then,

(i) $B = R$ if and only if

$$\sigma_B^{\pm}(y) = \sigma_R^{\pm}(y), \tag{22a}$$

$$\nu_B^{\pm}(y) = \nu_R^{\pm}(y), \tag{22b}$$

$$\gamma_B^{\pm}(y) = \gamma_R^{\pm}(y). \tag{22c}$$

(ii)

$$B \cup R = \{\langle y, \sigma^{M^+}(y), \nu^{m^+}(y), \gamma^{m^+}(y), \sigma^{m^-}(y), \nu^{M^-}(y), \gamma^{M^-}(y)\rangle\}. \tag{23}$$

where

$$\sigma^{M^+}(y) = \sigma_B^+(y) \vee \sigma_R^+(y), \tag{24a}$$

$$\nu^{m^+}(y) = \nu_B^+(y) \wedge \nu_R^+(y), \tag{24b}$$

$$\gamma^{m^+}(y) = \gamma_B^+(y) \wedge \gamma_R^+(y), \tag{24c}$$

$$\sigma^{m^-}(y) = \sigma_B^-(y) \wedge \sigma_R^-(y), \tag{24d}$$

$$\nu^{M^-}(y) = \nu_B^M(y) \vee \nu_R^-(y), \tag{24e}$$

$$\gamma^{M^-}(y) = \gamma_B^-(y) \vee \gamma_R^-(y)). \tag{24f}$$

(iii)

$$B \cap R = \{\langle y, \sigma^{m^+}(y), \nu^{M^+}(y), \gamma^{M^+}(y), \sigma^{M^-}(y), \nu^{m^-}(y), \gamma^{m^-}(y)\rangle\}. \tag{25}$$

where

$$\sigma^{m^+}(y) = \sigma_B^+(y) \wedge \sigma_R^+(y), \tag{26a}$$

$$\nu^{M^+}(y) = \nu_B^+(y) \vee \nu_R^+(y), \tag{26b}$$

$$\gamma^{M^+}(y) = \gamma_B^+(y) \vee \gamma_R^+(y), \tag{26c}$$

$$\sigma^{M^-}(y) = \sigma_B^-(y) \vee \sigma_R^-(y), \tag{26d}$$

$$\nu^{m^-}y) = \nu_B^M(y) \wedge \nu_R^-(y), \tag{26e}$$

$$\gamma^{m^-}(y) = \gamma_B^-(y) \wedge \gamma_R^-(y)). \tag{26f}$$

Definition 8: [8] Let $(G, \bullet)$ be a crisp group and $B = \{\sigma_B, \nu_B, \gamma_B\}$ be a PFSG of G. Then, for $a \in G$ the picture fuzzy left coset of $B \in G$ is the PFS $aB = \{\sigma_{aB},\ \nu_{aB},\ \gamma_{aB}\}$ defined by

$$\sigma_{aB}(u) = \sigma_B(a^{-1} \bullet u), \tag{27a}$$

$$\nu_{aB}(u) = \nu_B(a^{-1} \bullet u), \tag{27b}$$

$$\gamma_{aB}(u) = \gamma_B(a^{-1} \bullet u) \text{ for all } u \in G. \tag{27c}$$

Definition 9: [8] Let $(G, \bullet)$ be a crisp group and $B = \{\sigma_B, \nu_B, \gamma_B\}$ be a PFSG of G. Then, for $a \in G$ the picture fuzzy right coset of $B \in G$ is the PFS $Ba = \{\sigma_{Ba},\ \nu_{Ba},\ \gamma_{Ba}\}$ defined by

$$\sigma_{Ba}(u) = \sigma_B(u \bullet a^{-1}), \tag{28a}$$

$$\nu_{Ba}(u) = \nu_B(u \bullet a^{-1}), \tag{28b}$$

$$\gamma_{Ba}(u) = \gamma_B(u \bullet a^{-1}) \text{ for all } u \in G. \tag{28c}$$

Definition 10: [8] Let $(G, \bullet)$ be a crisp group and $B = \{\sigma_B, \nu_B, \gamma_B\}$ be a PFSG of G. Then, B is called picture fuzzy normal subgroup (PFNSG) of G if

$$\sigma_B(a \bullet b) = \sigma_B(b \bullet a), \tag{29a}$$

$$\nu_B(a \bullet b) = \nu_B(b \bullet a), \tag{29b}$$

$$\gamma_B(a \bullet b) = \gamma_B(b \bullet a) \text{ for all } a,\ b \in G \tag{29c}$$

Remark 2: Note that Definition 10 is equivalent to

$$\sigma_B(b) = \sigma_B(a^{-1}ba), \tag{30a}$$

$$\gamma_B(b) = \gamma_B(a^{-1}ba), \tag{30b}$$

$$\nu_B(b) = \nu_B(a^{-1}ba). \tag{30c}$$

3. SOME RESULTS ON BIPOLAR PICTURE FUZZY SUBGROUP (BPFSG)

Definition 11: Let $(G, \bullet)$ be a group. A bipolar picture fuzzy set (BPFS) B is called bipolar picture fuzzy subgroup (BPFSG) of G if for all $a, b \in G$

(i)

$$\sigma_B^+(a \bullet b) \geq \sigma_B^+(a) \wedge \sigma_B^+(b), \tag{31a}$$
$$\nu_B^+(a \bullet b) \geq \nu_B^+(a) \wedge \nu_B^+(b), \tag{31b}$$
$$\gamma_B^+(a \bullet b) \leq \gamma_B^+(a) \vee \gamma_B^+(b) \tag{31c}$$

(ii)

$$\sigma_B^-(a \bullet b) \leq \sigma_B^-(a) \vee \sigma_B^-(b), \tag{32a}$$
$$\nu_B^-(a \bullet b) \leq \nu_B^-(a) \vee \nu_B^-(b), \tag{32b}$$
$$\gamma_B^-(a \bullet b) \geq \gamma_B^-(a) \wedge \gamma_B^-(b) \tag{32c}$$

(iii)

$$\sigma_B^+(a^{-1}) \geq \sigma_B^+(a), \tag{33a}$$
$$\nu_B^+(a^{-1}) \geq \nu_B^+(a), \tag{33b}$$
$$\gamma_B^+(a^{-1}) \leq \gamma_B^+(a), \tag{33c}$$
$$\sigma_B^-(a^{-1}) \leq \sigma_B^-(a), \tag{33d}$$
$$\nu_B^-(a^{-1}) \leq \nu_B^-(a), \tag{33e}$$
$$\gamma_B^-(a^{-1}) \geq \gamma_B^-(a). \tag{33f}$$

Remark 3: It should be noted that this Definition 11 can easily be stated in terms of t-norm and t-conorm. Recall that the function

$$T : [0,1] \times [0,1] \to [0,1],$$

defined by

$$T(x,y) = \begin{cases} x, if\ x \leq y \\ y, if\ y \leq x, \end{cases} \tag{34}$$

is called the t-norm and the function

$$S : [0,1] \times [0,1] \to [0,1],$$

defined by

$$S(x,y) = \begin{cases} y, if\ x \leq y \\ x, if\ y \leq x, \end{cases} \tag{35}$$

is called the t-conorm. There is the need to define t-norm $\mathbb{T}$ and t-conorm $\mathbb{S}$, which satisfy the commutativity, associativity, monotone and neutral element properties, on the negative interval $[-1, 0]$

respectively as follows:

$$\mathbb{T} : [-1, 0] \times [-1, 0] \to [-1, 0],$$

defined by

$$\begin{aligned} \mathbb{T}(x, y) \quad &= -S(-x, -y) \\ &= -max\{-x, -y\} \\ &= min\{x, y\} \\ &= T(x, y) \end{aligned} \tag{36}$$

and

$$\mathbb{S} : [-1, 0] \times [-1, 0] \to [-1, 0],$$

defined by

$$\begin{aligned} \mathbb{S}(x, y) \quad &= -T(-x, -y) \\ &= -min\{-x, -y\} \\ &= max\{x, y\} \\ &= S(x, y) \end{aligned} \tag{37}$$

We only show that of $\mathbb{S}$.

(i) Commutativity:

$$\begin{aligned} \mathbb{S}(x, y) \quad &= -T(-x, -y) \\ &= S(x, y) \\ &= S(y, x) \\ &= -T(-y, -x) \\ &= \mathbb{S}(y, x) \end{aligned} \tag{38}$$

(ii) Associativity:

$$\begin{aligned} \mathbb{S}(\mathbb{S}(x, y), z) \quad &= \mathbb{S}(-T(-x, -y), z) \\ &= \mathbb{S}(S(x, y), z) \\ &= -T(-S(x, y), -z) \\ &= S(S(x, y), z) \\ &= S(x, S(y, z)) \\ &= -T(-x, -S(y, z)) \\ &= \mathbb{S}(x, S(y, z)) \\ &= \mathbb{S}(x, -T(-y, -z)) \\ &= \mathbb{S}(x, \mathbb{S}(y, z)) \end{aligned} \tag{39}$$

(iii) Monotone: Let $x \leq y$, then,

$$\begin{aligned} \mathbb{S}(x,z) &= -T(-x,-z) \\ &= S(x,z) \\ &\leq S(y,z) \\ &= -T(-y,-z) \\ &= \mathbb{S}(y,z) \end{aligned} \tag{40}$$

(iv) Neutral Element:

$$\begin{aligned} \mathbb{S}(x,0) &= -T(-x,0) \\ &= S(x,0) \\ &= 0 \end{aligned} \tag{41}$$

Hence, $\mathbb{T} = T$ and $\mathbb{S} = S$. As a result, the t-norm T and t-conorm S can be redefined to have an extended domain and codomain as follows:

$$T : [-1,1] \times [-1,1] \to [-1,1]$$

is defined by

$$T(x,y) = \begin{cases} x, if\ x \leq y \\ y, if\ y \leq x, \end{cases} \tag{42}$$

and

$$S : [-1,1] \times [-1,1] \to [-1,1]$$

is defined by

$$S(x,y) = \begin{cases} y, if\ x \leq y \\ x, if\ y \leq x, \end{cases} \tag{43}$$

The next definition is the version of Definition 11 in terms of t-norm and t-conorm.

Definition 12: Let $(G, \bullet)$ be a group. A bipolar picture fuzzy set (BPFS) B is called bipolar picture fuzzy subgroup (BPFSG) of G if for all $a, b \in G$

(i)

$$\sigma_B^+(a \bullet b) \geq T(\sigma_B^+(a), \sigma_B^+(b)) = \sigma_B^+(a) \wedge \sigma_B^+(b), \tag{44a}$$

$$\nu_B^+(a \bullet b) \geq T(\nu_B^+(a), \nu_B^+(b)) = \nu_B^+(a) \wedge \nu_B^+(b), \tag{44b}$$

$$\gamma_B^+(a \bullet b) \leq S(\gamma_B^+(a), \gamma_B^+(b)) = \gamma_B^+(a) \vee \gamma_B^+(b) \tag{44c}$$

(ii)

$$\sigma_B^-(a \bullet b) \leq S(\sigma_B^-(a), \sigma_B^-(b)) = \sigma_B^-(a) \vee \sigma_B^-(b), \tag{45a}$$

$$\nu_B^-(a \bullet b) \leq S(\nu_B^-(a), \nu_B^-(b)) = \nu_B^-(a) \vee \nu_B^-(b), \tag{45b}$$

$$\gamma_B^-(a \bullet b) \geq T(\gamma_B^-(a), \gamma_B^-(b)) = \gamma_B^-(a) \wedge \gamma_B^-(b) \tag{45c}$$

(iii) We do the following for T though a similar thing can be done for S. Since

$$\sigma_B^+(a^{-1}) \geq \sigma_B^+(a), T(\sigma_B^+(a^{-1}), \sigma_B^+(c)) \geq T(\sigma_B^+(a), \sigma_B^+(c)), \tag{46a}$$
$$\nu_B^+(a^{-1}) \geq \nu_B^+(a), T(\nu_B^+(a^{-1}), \nu_B^+(c)) \geq T(\nu_B^+(a), \nu_B^+(c)), \tag{46b}$$
$$\gamma_B^+(a^{-1}) \leq \gamma_B^+(a), T(\gamma_B^+(a^{-1}), \gamma_B^+(c)) \leq T(\gamma_B^+(a), \gamma_B^+(c)), \tag{46c}$$
$$\sigma_B^-(a^{-1}) \leq \sigma_B^-(a), T(\sigma_B^-(a^{-1}), \sigma_B^-(c)) \leq T(\sigma_B^-(a), \sigma_B^-(c)), \tag{46d}$$
$$\nu_B^-(a^{-1}) \leq \nu_B^-(a), T(\nu_B^-(a^{-1}), \nu_B^-(c)) \leq T(\nu_B^-(a), \nu_B^-(c)), \tag{46e}$$
$$\gamma_B^-(a^{-1}) \geq \gamma_B^-(a), T(\gamma_B^-(a^{-1}), \gamma_B^-(c)) \geq T(\gamma_B^-(a), \gamma_B^-(c)). \tag{46f}$$

Example 1: Let $G = \{1, -1, i, -i\}$ be the multiplicative group and $B = \left\{\langle a, \sigma_B^\pm(a), \nu_B^\pm(a), \gamma_B^\pm(a)\rangle\right\}$ be a BPFS whose memberships are defined by

$$\sigma_B^+(a) = \begin{cases} 0.6, & \text{if } a = 1 \\ 0.5, & \text{if } a = -1 \\ 0.4, & \text{if } a = -i, i \end{cases} \quad \sigma_B^-(a) = \begin{cases} -0.4, & \text{if } a = 1 \\ -0.3, & \text{if } a = -1 \\ -0.1, & \text{if } a = -i, i \end{cases} \tag{47}$$

$$\gamma_B^+(a) = \begin{cases} 0.15, & \text{if } a = 1 \\ 0.2, & \text{if } a = -1 \\ 0.3, & \text{if } a = -i, i \end{cases} \quad \gamma_B^-(a) = \begin{cases} -0.1, & \text{if } a = 1 \\ -0.2, & \text{if } a = -1 \\ -0.5, & \text{if } a = -i, i \end{cases} \tag{48}$$

$$\nu_B^+(a) = \begin{cases} 0.25, & \text{if } a = 1 \\ 0.2, & \text{if } a = -1 \\ 0.1, & \text{if } a = -i, i \end{cases} \quad \nu_B^-(a) = \begin{cases} -0.5, & \text{if } a = 1 \\ -0.4, & \text{if } a = -1 \\ -0.3, & \text{if } a = -i, i \end{cases} \tag{49}$$

Then B is a BPFSG of G.

Proposition 1: Let G be a group and

$$B = \left\{\langle a, \sigma_B^\pm(a), \nu_B^\pm(a), \gamma_B^\pm(a)\rangle\right\} \tag{50}$$

be BPFSG of G. Then,

(i)

$$\sigma_B^\pm(a^{-1}) = \sigma_B^\pm(a), \tag{51a}$$
$$\nu_B^\pm(a^{-1}) = \nu_B^\pm(a), \tag{51b}$$
$$\gamma_B^\pm(a^{-1}) = \gamma_B^\pm(a) \tag{51c}$$

(ii)

$$\sigma_B^+(e) \geq \sigma_B^+(a) \qquad \sigma_B^-(e) \leq \sigma_B^-(a) \tag{52}$$
$$\nu_B^+(e) \geq \nu_B^+(a) \qquad \nu_B^-(e) \leq \nu_B^-(a) \tag{53}$$
$$\gamma_B^+(e) \leq \gamma_B^+(a) \qquad \gamma_B^-(e) \geq \gamma_B^-(a) \tag{54}$$

for all $a \in G$, e is the identity element in G and a^{-1} is the inverse of $a \in G$.

Proof: Let $a, e \in G$.

(i) Since B is a BPFSG, then

$$\sigma_B^+(a^{-1}) \geq \sigma_B^+(a) \qquad \sigma_B^-(a^{-1}) \leq \sigma_B^-(a) \tag{55}$$
$$\nu_B^+(a^{-1}) \geq \nu_B^+(a) \qquad \nu_B^-(a^{-1}) \leq \nu_B^-(a) \tag{56}$$
$$\gamma_B^+(a^{-1}) \leq \gamma_B^+(a) \qquad \gamma_B^-(a^{-1}) \geq \gamma_B^-(a) \tag{57}$$

Also,

$$\sigma_B^+(a) = \sigma_B^+(a^{-1})^{-1} \geq \sigma_B^+(a^{-1}), \tag{58a}$$
$$\sigma_B^-(a) = \sigma_B^-(a^{-1})^{-1} \leq \sigma_B^-(a^{-1}) \tag{58b}$$

It can similarly be shown for ν and γ.

(ii)

$$\sigma_B^+(e) = \sigma_B^+(a \bullet a^{-1}) \geq \sigma_B^+(a), \tag{59a}$$
$$\sigma_B^-(e) = \sigma_B^-(a \bullet a^{-1}) \leq \sigma_B^-(a) \tag{59b}$$

Similarly, it can be shown for γ and ν.

Remark 4: Subsequently, the BPFSG B shall be defined as

$$B = \left\{ \langle a, \sigma_B^{\pm}(a), \nu_B^{\pm}(a), \gamma_B^{\pm}(a) \rangle \right\} \tag{60}$$

Proposition 2: If B is a BPFSG of G, then

(i)

$$\sigma_B^+(a \bullet b^{-1}) = \sigma_B^+(e) \implies \sigma_B^+(a) = \sigma_B^+(b), \tag{61a}$$
$$\sigma_B^-(a \bullet b^{-1}) = \sigma_B^-(e) \implies \sigma_B^-(a) = \sigma_B^-(b) \tag{61b}$$

(ii)

$$\nu_B^+(a \bullet b^{-1}) = \nu_B^+(e) \implies \nu_B^+(a) = \nu_B^+(b), \tag{62a}$$
$$\nu_B^-(a \bullet b^{-1}) = \nu_B^-(e) \implies \nu_B^-(a) = \nu_B^-(b) \tag{62b}$$

(iii)

$$\gamma_B^+(a \bullet b^{-1}) = \gamma_B^+(e) \implies \gamma_B^+(a) = \gamma_B^+(b), \tag{63a}$$
$$\gamma_B^-(a \bullet b^{-1}) = \gamma_B^-(e) \implies \gamma_B^-(a) = \gamma_B^-(b) \tag{63b}$$

for $a, b, e \in G$.
Proof: Let $a, b, e \in G$.

(i)

$$\sigma_B^+(a) = \sigma_B^+(a \bullet b^{-1} \bullet b) \geq \sigma_B^+(b), \tag{64a}$$
$$\sigma_B^+(b) = \sigma_B^+(b \bullet a^{-1} \bullet a) \geq \sigma_B^+(a), \tag{64b}$$
$$\sigma_B^-(a) = \sigma_B^-(a \bullet b^{-1} \bullet b) \leq \sigma_B^-(b), \tag{64c}$$
$$\sigma_B^-(b) = \sigma_B^-(b \bullet a^{-1} \bullet a) \leq \sigma_B^-(a) \tag{64d}$$

Hence,

$$\sigma_B^+(a) = \sigma_B^+(b) \qquad \sigma_B^-(a) = \sigma_B^-(b). \tag{65}$$

(ii) The proof is similar to (i).
(iii) The proof is similar to (i).

Proposition 3: Let G be a group, $a, b \in G$ and B a BPFS. Then, B is a BPFSG of G if and only if

$$\sigma_B^+(a \bullet b^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b), \tag{66a}$$
$$\nu_B^+(a \bullet b^{-1}) \geq \nu_B^+(a) \wedge \nu_B^+(b), \tag{66b}$$
$$\gamma_B^+(a \bullet b^{-1}) \leq \gamma_B^+(a) \vee \gamma_B^+(b) \tag{66c}$$

and

$$\sigma_B^-(a \bullet b^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b), \tag{67a}$$
$$\nu_B^-(a \bullet b^{-1}) \leq \nu_B^-(a) \vee \nu_B^-(b), \tag{67b}$$
$$\gamma_B^-(a \bullet b^{-1}) \geq \gamma_B^-(a) \wedge \gamma_B^-(b). \tag{67c}$$

Proof: Suppose that B is a BPFSG of G. Then, by Proposition 1

$$\sigma_B^+(a \bullet b^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b), \tag{68a}$$
$$\sigma_B^-(a \bullet b^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b), \tag{68b}$$
$$\nu_B^+(a \bullet b^{-1}) \geq \nu_B^+(a) \wedge \nu_B^+(b), \tag{68c}$$
$$\nu_B^-(a \bullet b^{-1}) \leq \nu_B^-(a) \vee \nu_B^-(b) \tag{68d}$$
$$\gamma_B^+(a \bullet b^{-1}) \leq \gamma_B^+(a) \vee \gamma_B^+(b), \tag{68e}$$
$$\gamma_B^-(a \bullet b^{-1}) \geq \gamma_B^-(a) \wedge \gamma_B^-(b). \tag{68f}$$

Conversely, if

$$\sigma_B^+(a \bullet b^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b), \tag{69a}$$
$$\nu_B^+(a \bullet b^{-1}) \geq \nu_B^+(a) \wedge \nu_B^+(b), \tag{69b}$$
$$\gamma_B^+(a \bullet b^{-1}) \leq \gamma_B^+(a) \vee \gamma_B^+(b) \tag{69c}$$

and

$$\sigma_B^-(a \bullet b^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b), \tag{70a}$$
$$\nu_B^-(a \bullet b^{-1}) \leq \nu_B^-(a) \vee \nu_B^-(b), \tag{70b}$$
$$\gamma_B^-(a \bullet b^{-1}) \geq \gamma_B^-(a) \wedge \gamma_B^-(b). \tag{70c}$$

Replace b by a and proving for σ, while that of ν and γ can similarly be shown,

$$\sigma_B^+(a) \leq \sigma_B^+(e), \sigma_B^-(a) \geq \sigma_B^-(e) \tag{71}$$

for all $a, b \in G$.
Now,

$$\sigma_B^+(a^{-1}) = \sigma_B^+(e \bullet a^{-1}) \geq \sigma_B^+(e) \wedge \sigma_B^+(a) = \sigma_B^+(a) \tag{72a}$$
$$\sigma_B^-(a^{-1}) = \sigma_B^-(e \bullet a^{-1}) \leq \sigma_B^-(e) \vee \sigma_B^-(a) = \sigma_B^-(a) \tag{72b}$$

It follows that,

$$\sigma_B^+(a \bullet b) = \sigma_B^+(a \bullet (b^{-1})^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b) \tag{73}$$

$$\sigma_B^-(a \bullet b) = \sigma_B^-(a \bullet (b^{-1})^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b). \tag{74}$$

Hence, B is a BPFSG of G.
Proposition 4: Let B be a BPFSG of G and $a \in G$. Then,

$$\sigma_B^\pm(a \bullet b) = \sigma_B^\pm(b), \tag{75a}$$
$$\nu_B^\pm(a \bullet b) = \nu_B^\pm(b), \tag{75b}$$
$$\gamma_B^\pm(a \bullet b) = \gamma_B^\pm(b) \tag{75c}$$

for every $b \in G$ if and only if

$$\sigma_B^\pm(a) = \sigma_B^\pm(e), \tag{76a}$$
$$\nu_B^\pm(a) = \nu_B^\pm(e), \tag{76b}$$
$$\gamma_B^\pm(a) = \gamma_B^\pm(e) \tag{76c}$$

Proof: Suppose that

$$\sigma_B^\pm(a \bullet b) = \sigma_B^\pm(b), \tag{77a}$$
$$\nu_B^\pm(a \bullet b) = \nu_B^\pm(b), \tag{77b}$$
$$\gamma_B^\pm(a \bullet b) = \gamma_B^\pm(b) \tag{77c}$$

for every $b \in G$. When $b = e$, then

$$\sigma_B^{\pm}(a) = \sigma_B^{\pm}(e), \tag{78a}$$
$$\nu_B^{\pm}(a) = \nu_B^{\pm}(e), \tag{78b}$$
$$\gamma_B^{\pm}(a) = \gamma_B^{\pm}(e) \tag{78c}$$

Conversely, Suppose that

$$\sigma_B^{\pm}(a) = \sigma_B^{\pm}(e), \tag{79a}$$
$$\nu_B^{\pm}(a) = \nu_B^{\pm}(e), \tag{79b}$$
$$\gamma_B^{\pm}(a) = \gamma_B^{\pm}(e) \tag{79c}$$

While the proof for ν and γ are left, that of σ is shown because they are similar. Thus, since B is a BPFSG and by Proposition 1,

$$\sigma_B^{+}(a \bullet b) \geq \sigma_B^{+}(a) \wedge \sigma_B^{+}(b) = \sigma_B^{+}(b), \tag{80a}$$
$$\sigma_B^{-}(a \bullet b) \leq \sigma_B^{-}(a) \vee \sigma_B^{-}(b) = \sigma_B^{-}(b), \tag{80b}$$

for all $b \in G$.
Also,

$$\sigma_B^{+}(b) = \sigma_B^{+}(a^{-1} \bullet a \bullet b) \tag{81}$$
$$= \sigma_B^{+}(a^{-1} \bullet (a \bullet b)) \tag{82}$$
$$\geq \sigma_B^{+}(a^{-1}) \wedge \sigma_B^{+}(a \bullet b) \tag{83}$$
$$\geq \sigma_B^{+}(a) \wedge \sigma_B^{+}(a \bullet b) \tag{84}$$
$$= \sigma_B^{+}(e) \wedge \sigma_B^{+}(a \bullet b) \tag{85}$$
$$= \sigma_B^{+}(a \bullet b) \tag{86}$$

$$\sigma_B^{-}(b) = \sigma_B^{-}(a^{-1} \bullet a \bullet b) \tag{87}$$
$$= \sigma_B^{-}(a^{-1} \bullet (a \bullet b)) \tag{88}$$
$$\leq \sigma_B^{-}(a^{-1}) \vee \sigma_B^{-}(a \bullet b) \tag{89}$$
$$\leq \sigma_B^{-}(a) \vee \sigma_B^{-}(a \bullet b) \tag{90}$$
$$= \sigma_B^{-}(e) \vee \sigma_B^{-}(a \bullet b) \tag{91}$$
$$= \sigma_B^{-}(a \bullet b) \tag{92}$$

Therefore,

$$\sigma_B^{\pm}(b) = \sigma_B^{\pm}(a \bullet b), \tag{93a}$$
$$\nu_B^{\pm}(b) = \nu_B^{\pm}(a \bullet b), \tag{93b}$$
$$\gamma_B^{\pm}(b) = \gamma_B^{\pm}(a \bullet b) \tag{93c}$$

for all $b \in G$.

Proposition 5: Let B and $R = \left\{\sigma_R^+, \sigma_R^-, \nu_R^+, \nu_R^-, \gamma_R^+, \gamma_R^-\right\}$ be two BPFSGs of G. Then, $B \cap R$ is a BPFSG of G.
Proof: Let $B \cap R = S = \left\{\sigma_S^+, \sigma_S^-, \nu_S^+, \nu_S^-, \gamma_S^+, \gamma_S^-\right\}$ such that

$$\sigma_S^+(a) = \sigma_B^+(a) \wedge \sigma_R^+(a), \tag{94a}$$
$$\nu_S^+(a) = \nu_B^+(a) \vee \nu_R^+(a), \tag{94b}$$
$$\gamma_S^+(a) = \gamma_B^+(a) \vee \gamma_R^+(a) \tag{94c}$$

and

$$\sigma_S^-(a) = \sigma_B^-(a) \vee \sigma_R^-(a), \tag{95a}$$
$$\nu_S^-(a) = \nu_B^-(a) \wedge \nu_R^-(a), \tag{95b}$$
$$\gamma_S^-(a) = \gamma_B^-(a) \wedge \gamma_R^-(a), \tag{95c}$$

for all $a \in G$.

The proof will be done for σ and those of ν and γ can be similarly shown. Since B, R are BPFSGs of G then,

$$\sigma_S^+(a \bullet b^{-1}) = \sigma_B^+(a \bullet b^{-1}) \wedge \sigma_R^+(a \bullet b^{-1}) \tag{96}$$
$$\geq (\sigma_B^+(a) \wedge \sigma_R^+(a)) \wedge (\sigma_B^+(b) \wedge \sigma_R^+(b)) \tag{97}$$
$$= \sigma_S^+(a) \wedge \sigma_S^+(b) \tag{98}$$

and

$$\sigma_S^-(a \bullet b^{-1}) = \sigma_B^-(a \bullet b^{-1}) \vee \sigma_R^-(a \bullet b^{-1}) \tag{99}$$
$$\leq (\sigma_B^-(a) \vee \sigma_R^-(a)) \vee (\sigma_B^-(b) \vee \sigma_R^-(b)) \tag{100}$$
$$= \sigma_S^-(a) \vee \sigma_S^-(b) \tag{101}$$

for all $a, b \in G$. Hence, $S = B \cap R$ is a BPFSG of G.
Example 2: Let $G = \{1, -1, i, -i\}$ be the multiplicative group, $B = \left\{\langle a, \sigma_B^{\pm}(a), \nu_B^{\pm}(a), \gamma_B^{\pm}(a)\rangle\right\}$ and $R = \left\{\langle a, \sigma_R^{\pm}(a), \nu_R^{\pm}(a), \gamma_R^{\pm}(a)\rangle\right\}$ be BPFSGs of G whose memberships are respectively defined by

$$\sigma_B^+(a) = \begin{cases} 0.6, \text{ if } a = 1 \\ 0.5, \text{ if } a = -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma_B^-(a) = \begin{cases} -0.4, \text{ if } a = 1 \\ -0.3, \text{ if } a = -1 \\ -0.1, \text{ if } a = -i, i \end{cases} \tag{102}$$

$$\gamma_B^+(a) = \begin{cases} 0.15, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.3, \text{ if } a = -i, i \end{cases} \quad \gamma_B^-(a) = \begin{cases} -0.1, \text{ if } a = 1 \\ -0.2, \text{ if } a = -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{103}$$

$$\nu_B^+(a) = \begin{cases} 0.25, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu_B^-(a) = \begin{cases} -0.5, \text{ if } a = 1 \\ -0.4, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{104}$$

and

$$\sigma_R^+(a) = \begin{cases} 0.62, \text{ if } a = 1 \\ 0.45, \text{ if } a = -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma_R^-(a) = \begin{cases} -0.38, \text{ if } a = 1 \\ -0.29, \text{ if } a = -1 \\ -0.1, \text{ if } a = -i, i \end{cases} \tag{105}$$

$$\gamma_R^+(a) = \begin{cases} 0.13, \text{ if } a = 1 \\ 0.25, \text{ if } a = -1 \\ 0.26, \text{ if } a = -i, i \end{cases} \quad \gamma_R^-(a) = \begin{cases} -0.12, \text{ if } a = 1 \\ -0.21, \text{ if } a = -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{106}$$

$$\nu_R^+(a) = \begin{cases} 0.22, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu_R^-(a) = \begin{cases} -0.49, \text{ if } a = 1 \\ -0.35, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{107}$$

Then $S = B \cap R$ defined by

$$\sigma_S^+(a) = \begin{cases} 0.6, \text{ if } a = 1 \\ 0.45, \text{ if } a = -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma_S^-(a) = \begin{cases} -0.38, \text{ if } a = 1 \\ -0.29, \text{ if } a = -1 \\ -0.1, \text{ if } a = -i, i \end{cases} \tag{108}$$

$$\gamma_S^+(a) = \begin{cases} 0.15, \text{ if } a = 1 \\ 0.25, \text{ if } a = -1 \\ 0.3, \text{ if } a = -i, i \end{cases} \quad \gamma_S^-(a) = \begin{cases} -0.12, \text{ if } a = 1 \\ -0.21, \text{ if } a = -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{109}$$

$$\nu_S^+(a) = \begin{cases} 0.25, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu_S^-(a) = \begin{cases} -0.5, \text{ if } a = 1 \\ -0.4, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{110}$$

is a BPFSG of G.

Proposition 6: Let B and $R = \left\{\sigma_R^+, \sigma_R^-, \nu_R^+, \nu_R^-, \gamma_R^+, \gamma_R^-\right\}$ be two BPFSGs of G. Then, $B \cup R$ is a BPFSG of G if $B \subseteq R$ or $R \subseteq B$.

Proof: Let

$$B \cup R = W = \left\{\sigma_W^+, \sigma_W^-, \nu_W^+, \nu_W^-, \gamma_W^+, \gamma_W^-\right\} \tag{111}$$

such that

$$\sigma_W^+(a) = \sigma_B^+(a) \vee \sigma_R^+(a), \tag{112a}$$
$$\nu_W^+(a) = \nu_B^+(a) \wedge \nu_R^+(a), \tag{112b}$$
$$\gamma_W^+(a) = \gamma_B^+(a) \wedge \gamma_R^+(a) \tag{112c}$$

and

$$\sigma_W^-(a) = \sigma_B^-(a) \wedge \sigma_R^-(a), \tag{113a}$$
$$\nu_W^-(a) = \nu_B^-(a) \vee \nu_R^-(a), \tag{113b}$$
$$\gamma_W^-(a) = \gamma_B^-(a) \vee \gamma_R^-(a), \tag{113c}$$

for all a $\in G$.

The proof will be done for σ.

Case 1. Let $B \subseteq R$. Then,

$$\sigma_B^+(a) \leq \sigma_R^+(a), \qquad \sigma_B^-(a) \geq \sigma_R^-(a) \tag{114}$$

for all $a \in G$. Now,

$$\sigma_W^+(a \bullet b^{-1}) = \sigma_B^+(a \bullet b^{-1}) \vee \sigma_R^+(a \bullet b^{-1}) \tag{115}$$
$$= \sigma_R^+(a \bullet b^{-1}) \geq \sigma_R^+(a) \wedge \sigma_R^+(b)) \tag{116}$$
$$= (\sigma_B^+(a) \vee \sigma_R^+(a)) \wedge (\sigma_B^+(b) \vee \sigma_R^+(b)) \tag{117}$$
$$= \sigma_W^+(a) \wedge \sigma_W^+(b) \tag{118}$$

and

$$\sigma_W^-(a \bullet b^{-1}) = \sigma_B^-(a \bullet b^{-1}) \wedge \sigma_R^-(a \bullet b^{-1}) \tag{119}$$
$$= \sigma_R^-(a \bullet b^{-1}) \leq \sigma_R^-(a) \vee \sigma_R^-(b)) \tag{120}$$
$$= (\sigma_B^-(a) \wedge \sigma_R^-(a)) \vee (\sigma_B^-(b) \wedge \sigma_R^-(b)) \tag{121}$$
$$= \sigma_W^-(a) \vee \sigma_W^-(b) \tag{122}$$

for all $a, b \in G$. Hence, $W = B \cup R$ is a BPFSG of G.

Similarly, Case 2 can be shown by following the procedure for Case 1.

Example 3: Let $G = \{1, -1, i, -i\}$ be the multiplicative group, $B = \left\{\langle a, \sigma_B^{\pm}(a), \nu_B^{\pm}(a), \gamma_B^{\pm}(a)\rangle\right\}$ and $R = \left\{\langle a, \sigma_R^{\pm}(a), \nu_R^{\pm}(a), \gamma_R^{\pm}(a)\rangle\right\}$ be BPFSGs of G whose memberships are respectively defined by

$$\sigma_B^+(a)=\begin{cases}0.5, \text{ if } a=1\\ 0.42, \text{ if } a=-1\\ 0.4, \text{ if } a=-i,i\end{cases} \quad \sigma_B^-(a)=\begin{cases}-0.38, \text{ if } a=1\\ -0.29, \text{ if } a=-1\\ -0.1, \text{ if } a=-i,i\end{cases} \tag{123}$$

$$\gamma_B^+(a)=\begin{cases}0.15, \text{ if } a=1\\ 0.25, \text{ if } a=-1\\ 0.3, \text{ if } a=-i,i\end{cases} \quad \gamma_B^-(a)=\begin{cases}-0.12, \text{ if } a=1\\ -0.21, \text{ if } a=-1\\ -0.5, \text{ if } a=-i,i\end{cases} \tag{124}$$

$$\nu_B^+(a)=\begin{cases}0.22, \text{ if } a=1\\ 0.2, \text{ if } a=-1\\ 0.1, \text{ if } a=-i,i\end{cases} \quad \nu_B^-(a)=\begin{cases}-0.49, \text{ if } a=1\\ -0.35, \text{ if } a=-1\\ -0.3, \text{ if } a=-i,i\end{cases} \tag{125}$$

and

$$\sigma_R^+(a)=\begin{cases}0.52, \text{ if } a=1\\ 0.45, \text{ if } a=-1\\ 0.4, \text{ if } a=-i,i\end{cases} \quad \sigma_R^-(a)=\begin{cases}-0.4, \text{ if } a=1\\ -0.3, \text{ if } a=-1\\ -0.2, \text{ if } a=-i,i\end{cases} \tag{126}$$

$$\gamma_R^+(a)=\begin{cases}0.13, \text{ if } a=1\\ 0.2, \text{ if } a=-1\\ 0.28, \text{ if } a=-i,i\end{cases} \quad \gamma_R^-(a)=\begin{cases}-0.1, \text{ if } a=1\\ -0.2, \text{ if } a=-1\\ -0.5, \text{ if } a=-i,i\end{cases} \tag{127}$$

$$\nu_R^+(a)=\begin{cases}0.25, \text{ if } a=1\\ 0.2, \text{ if } a=-1\\ 0.1, \text{ if } a=-i,i\end{cases} \quad \nu_R^-(a)=\begin{cases}-0.5, \text{ if } a=1\\ -0.4, \text{ if } a=-1\\ -0.3, \text{ if } a=-i,i\end{cases} \tag{128}$$

Then $W=B\cup R$ defined by

$$\sigma_W^+(a)=\begin{cases}0.52, \text{ if } a=1\\ 0.45, \text{ if } a=-1\\ 0.4, \text{ if } a=-i,i\end{cases} \quad \sigma_W^-(a)=\begin{cases}-0.4, \text{ if } a=1\\ -0.3, \text{ if } a=-1\\ -0.2, \text{ if } a=-i,i\end{cases} \tag{129}$$

$$\gamma_W^+(a)=\begin{cases}0.13, \text{ if } a=1\\ 0.2, \text{ if } a=-1\\ 0.28, \text{ if } a=-i,i\end{cases} \quad \gamma_W^-(a)=\begin{cases}-0.1, \text{ if } a=1\\ -0.2, \text{ if } a=-1\\ -0.5, \text{ if } a=-i,i\end{cases} \tag{130}$$

$$\nu_W^+(a) = \begin{cases} 0.22, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu_W^-(a) = \begin{cases} -0.49, \text{ if } a = 1 \\ -0.35, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{131}$$

is a BPFSG of G.
Proposition 7: Let B be a BPFSG of G. Then, the set H such that

$$H = \left\{a \in G | \sigma_B^{\pm}(a) = \sigma_B^{\pm}(e), \nu_B^{\pm}(a) = \nu_B^{\pm}(e), \gamma_B^{\pm}(a) = \gamma_B^{\pm}(e)\right\}$$

is a subgroup of G.
Proof: Let H be as defined above and $a, b \in G$. By Proposition 1,

$$\sigma_B^{\pm}(b^{-1}) = \sigma_B^{\pm}(b) = \sigma_B^{\pm}(e), \tag{132a}$$
$$\nu_B^{\pm}(b^{-1}) = \nu_B^{\pm}(b) = \nu_B^{\pm}(e), \tag{132b}$$
$$\gamma_B^{\pm}(b^{-1}) = \gamma_B^{\pm}(b) = \gamma_B^{\pm}(e). \tag{132c}$$

Therefore, $b^{-1} \in H$.
Now,

$$\sigma_B^+(a \bullet b^{-1}) \geq \sigma_B^+(a) \wedge \sigma_B^+(b^{-1}) = \sigma_B^+(e), \tag{133a}$$
$$\sigma_B^-(a \bullet b^{-1}) \leq \sigma_B^-(a) \vee \sigma_B^-(b^{-1}) = \sigma_B^-(e), \tag{133b}$$
$$\nu_B^+(a \bullet b^{-1}) \geq \nu_B^+(a) \wedge \nu_B^+(b^{-1}) = \nu_B^+(e), \tag{133c}$$
$$\nu_B^-(a \bullet b^{-1}) \leq \nu_B^-(a) \vee \nu_B^-(b^{-1}) = \nu_B^-(e), \tag{133d}$$
$$\gamma_B^+(a \bullet b^{-1}) \leq \gamma_B^+(a) \vee \gamma_B^+(b^{-1}) = \gamma_B^+(e), \tag{133e}$$
$$\gamma_B^-(a \bullet b^{-1}) \geq \gamma_B^-(a) \wedge \gamma_B^-(b^{-1}) = \gamma_B^-(e). \tag{133f}$$

Also,

$$\sigma_B^+(e) = \sigma_B^+((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \geq \sigma_B^+(a \bullet b^{-1}), \tag{134a}$$
$$\sigma_B^-(e) = \sigma_B^-((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \leq \sigma_B^-(a \bullet b^{-1}), \tag{134b}$$
$$\nu_B^+(e) = \nu_B^+((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \geq \nu_B^+(a \bullet b^{-1}), \tag{134c}$$
$$\nu_B^-(e) = \nu_B^-((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \leq \nu_B^-(a \bullet b^{-1}), \tag{134d}$$
$$\gamma_B^+(e) = \gamma_B^+((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \leq \gamma_B^+(a \bullet b^{-1}), \tag{134e}$$
$$\gamma_B^-(e) = \gamma_B^-((a \bullet b^{-1}) \bullet (a \bullet b^{-1})^{-1}) \geq \gamma_B^-(a \bullet b^{-1}). \tag{134f}$$

Hence, from (133) and (134),

$$\sigma_B^{\pm}(e) = \sigma_B^{\pm}(a \bullet b^{-1}), \tag{135a}$$
$$\nu_B^{\pm}(e) = \nu_B^{\pm}(a \bullet b^{-1}), \tag{135b}$$
$$\gamma_B^{\pm}(e) = \gamma_B^{\pm}(a \bullet b^{-1}). \tag{135c}$$

Therefore, $a \bullet b^{-1} \in H$ and H is a subgroup of G.

Definition 13: Let G be a group. A BPFSG B of G is called a bipolar picture fuzzy normal subgroup (BPFNSG) of G if

$$\sigma_B^{\pm}(a \bullet b) = \sigma_B^{\pm}(b \bullet a), \tag{136a}$$
$$\nu_B^{\pm}(a \bullet b) = \nu_B^{\pm}(b \bullet a), \tag{136b}$$
$$\gamma_B^{\pm}(a \bullet b) = \gamma_B^{\pm}(b \bullet a) \tag{136c}$$

for all $a, b \in G$.

Definition 14: Let G be a group. A BPFSG B of G is called a bipolar picture fuzzy normal subgroup (BPFNSG) of G if

$$\sigma_B^{\pm}(a \bullet b) = \sigma_B^{\pm}(b \bullet a), \tag{137a}$$
$$\nu_B^{\pm}(a \bullet b) = \nu_B^{\pm}(b \bullet a), \tag{137b}$$
$$\gamma_B^{\pm}(a \bullet b) = \gamma_B^{\pm}(b \bullet a) \tag{137c}$$

for all $a, b \in G$.

Definition 15: Let B be a BPFSG of a group G and H be as defined in Proposition 7. Then, the order of B, denoted as $o(B)$, is such that $o(B) = o(H)$.

Proposition 8: Let B be a BPFSG of G, then $o(B) \mid o(G)$.

Proof: Let B be a BPFSG of G and with identity e. Clearly, by Proposition , H is a subgroup of G. By Lagrange's theorem, $o(H) \mid o(G)$. Hence by Definition , $o(B) = o(H)$ and $o(B) \mid o(G)$.

Example 4: Let $G = \{1, -1, i, -i\}$ be the multiplicative group and B be defined by

$$\sigma_B^{+}(a) = \begin{cases} 0.6, \text{ if } a = 1, -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma_B^{-}(a) = \begin{cases} -0.4, \text{ if } a = 1, -1 \\ -0.1, \text{ if } a = -i, i \end{cases} \tag{138}$$

$$\gamma_B^{+}(a) = \begin{cases} 0.15, \text{ if } a = 1, -1 \\ 0.3, \text{ if } a = -i, i \end{cases} \quad \gamma_B^{-}(a) = \begin{cases} -0.1, \text{ if } a = 1, -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{139}$$

$$\nu_B^+(a) = \begin{cases} 0.25, \text{ if } a = 1, -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu_B^-(a) = \begin{cases} -0.5, \text{ if } a = 1, -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{140}$$

is a BPFSG of G. $H = \{1, -1\}$ is a subgroup of G and $o(H) = 2 = o(B)$. Furthermore, since G is abelian, the B is a BPNFSG of G

Definition 16: Let A and B be two BPFSGs of G. Then, A is conjugate to B if there exists $a \in G$ such that

$$\sigma_A^\pm(u) = \sigma_B^\pm(a^{-1} \bullet u \bullet a), \tag{141a}$$

$$\nu_A^\pm(u) = \nu_B^\pm(a^{-1} \bullet u \bullet a), \tag{141b}$$

$$\gamma_A^\pm(u) = \gamma_B^\pm(a^{-1} \bullet u \bullet a) \tag{141c}$$

for all $u \in G$.

Example 5: Let G and B be as in Example 4. B is self-conjugate.

Proposition 9: Two bipolar picture fuzzy subsets A and B of abelian group G are conjugate bipolar picture fuzzy subsets of G if and only if $A = B$.

Proof: Let A and B be conjugate bipolar picture fuzzy subsets of G, then for some $b \in G$,

$$\sigma_A^\pm(u) = \sigma_B^\pm(b^{-1} \bullet u \bullet b) = \sigma_B^\pm(e \bullet u) = \sigma_B^\pm(u), \tag{142a}$$

$$\nu_A^\pm(u) = \nu_B^\pm(b^{-1} \bullet u \bullet b) = \nu_B^\pm(e \bullet u) = \nu_B^\pm(u), \tag{142b}$$

$$\gamma_A^\pm(u) = \gamma_B^\pm(b^{-1} \bullet u \bullet b) = \gamma_B^\pm(e \bullet u) = \gamma_B^\pm(u). \tag{142c}$$

Therefore, $A = B$.

Conversely, Suppose that $A = B$, then for $e \in G$,

$$\sigma_A^\pm(u) = \sigma_B^\pm(e^{-1} \bullet u \bullet e), \tag{143a}$$

$$\nu_A^\pm(u) = \nu_B^\pm(e^{-1} \bullet u \bullet e), \tag{143b}$$

$$\gamma_A^+(u) = \gamma_B^+(e^{-1} \bullet u \bullet e) \tag{143c}$$

for every $u \in G$. Hence, A and B are conjugate bipolar picture fuzzy subsets of G.

Corollary 1: Any BPFS of an abelian group is self-conjugate.

Proposition 10: If A and B, as described in Proposition 9, are conjugate bipolar picture fuzzy subgroups of G, then $o(A) = o(B)$.

Proof: Let A and B be conjugate bipolar picture fuzzy subgroups of G. Now

$$o(A) = o(\{a \in G : \sigma_A^\pm(a) = \sigma_A^\pm(e), \nu_A^\pm(a) = \nu_A^\pm(e), \gamma_A^\pm(a) = \gamma_A^\pm(e)\}) \tag{144}$$

so that

$$o(A) = o(\{a \in G : \sigma_B^{\pm}(b^{-1} \bullet a \bullet b) = \sigma_B^{\pm}(b^{-1} \bullet e \bullet b), \quad (145)$$
$$\nu_B^{\pm}(b^{-1} \bullet a \bullet b) = \nu_B^{\pm}(b^{-1} \bullet e \bullet b), \quad (146)$$
$$\gamma_B^{\pm}(b^{-1} \bullet a \bullet b) = \gamma_B^{\pm}(b^{-1} \bullet e \bullet b)\}) \quad (147)$$

and

$$o(A) = o(\{a \in G : \sigma_B(a) = \sigma_B(e), \nu_B(a) \quad (148)$$
$$= \nu_B(e), \gamma_B^{\pm}(a) = \gamma_B^{\pm}(e)\}) = o(B). \quad (149)$$

Hence, $o(A) = o(B)$.
Proposition 11: Let A be a BPFSG of G and B a BPFS of G. If A and B are conjugates then B is a BPFSG of G.
Proof: Let A and B be conjugates, then for $b \in G$,

$$\sigma_B^{\pm}(u) = \sigma_A^{\pm}(b^{-1} \bullet u \bullet b), \quad (150a)$$
$$\nu_B^{\pm}(u) = \nu_A^{\pm}(b^{-1} \bullet u \bullet b), \quad (150b)$$
$$\gamma_B^{\pm}(u) = \gamma_A^{\pm}(b^{-1} \bullet u \bullet b) \quad (150c)$$

Thus,

$$\sigma_B^{+}(u \bullet v^{-1}) = \sigma_A^{+}(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \quad (151)$$
$$= \sigma_A^{+}(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \quad (152)$$
$$\geq \sigma_A^{+}(b^{-1} \bullet u \bullet b) \wedge \sigma_A^{+}(b^{-1} \bullet v^{-1} \bullet b) \quad (153)$$
$$= \sigma_B^{+}(u) \wedge \sigma_B^{+}(v) \quad (154)$$

$$\sigma_B^{-}(u \bullet v^{-1}) = \sigma_A^{-}(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \quad (155)$$
$$= \sigma_A^{-}(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \quad (156)$$
$$\leq \sigma_A^{-}(b^{-1} \bullet u \bullet b) \vee \sigma_A^{-}(b^{-1} \bullet v^{-1} \bullet b) \quad (157)$$
$$= \sigma_B^{-}(u) \vee \sigma_B^{+}(v) \quad (158)$$

$$\nu_B^{+}(u \bullet v^{-1}) = \nu_A^{+}(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \quad (159)$$
$$= \nu_A^{+}(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \quad (160)$$
$$\geq \nu_A^{+}(b^{-1} \bullet u \bullet b) \wedge \nu_A^{+}(b^{-1} \bullet v^{-1} \bullet b) \quad (161)$$
$$= \nu_B^{+}(u) \wedge \nu_B^{+}(v) \quad (162)$$

$$\nu_B^-(u \bullet v^{-1}) = \nu_A^-(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \tag{163}$$
$$= \nu_A^-(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \tag{164}$$
$$\leq \nu_A^-(b^{-1} \bullet u \bullet b) \vee \nu_A^-(b^{-1} \bullet v^{-1} \bullet b) \tag{165}$$
$$= \nu_B^-(u) \vee \nu_B^+(v) \tag{166}$$

$$\gamma_B^+(u \bullet v^{-1}) = \gamma_A^+(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \tag{167}$$
$$= \gamma_A^+(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \tag{168}$$
$$\leq \gamma_A^+(b^{-1} \bullet u \bullet b) \vee \gamma_A^+(b^{-1} \bullet v^{-1} \bullet b) \tag{169}$$
$$= \gamma_B^+(u) \vee \gamma_B^+(v) \tag{170}$$

$$\gamma_B^-(u \bullet v^{-1}) = \gamma_A^-(b^{-1} \bullet (u \bullet v^{-1}) \bullet b) \tag{171}$$
$$= \gamma_A^-(b^{-1} \bullet u \bullet b \bullet b^{-1} \bullet v^{-1} \bullet b) \tag{172}$$
$$\geq \gamma_A^-(b^{-1} \bullet u \bullet b) \wedge \gamma_A^-(b^{-1} \bullet v^{-1} \bullet b) \tag{173}$$
$$= \gamma_B^-(u) \wedge \gamma_B^+(v) \tag{174}$$

Therefore, B is a BPFSG of G.
Proposition 12: Let B be a BPFSG. If G is abelian group, then B is a BPFNSG of G.
Proof: Let G be abelian, then $a \bullet b = b \bullet a$ for all $a, b \in G$. Clearly, for any BPFSG B of G,

$$\sigma_B^{\pm}(a \bullet b) = \sigma_B^{\pm}(b \bullet a), \tag{175a}$$
$$\nu_B^{\pm}(a \bullet b) = \nu_B^{\pm}(b \bullet a), \tag{175b}$$
$$\gamma_B^{\pm}(a \bullet b) = \gamma_B^{\pm}(b \bullet a) \tag{175c}$$

Hence, B is BPFNSG.

4. SOME PROPERTIES OF BIPOLAR PICTURE FUZZY COSETS

Definition 17: Let B be a BPFSG of a group G. For any $a \in G$, the bipolar picture fuzzy left coset (BPFLC) is the set

$$aB(u) = \left\{\sigma_{aB}^{\pm}(u), \nu_{aB}^{\pm}(u), \gamma_{aB}^{\pm}(u)\right\} \tag{176}$$

such that

$$\sigma^{\pm}_{aB}(u) = \sigma^{\pm}_{B}(a^{-1} \bullet u), \tag{177a}$$
$$\nu^{\pm}_{aB}(u) = \nu^{\pm}_{B}(a^{-1} \bullet u), \tag{177b}$$
$$\gamma^{\pm}_{aB}(u) = \gamma^{\pm}_{B}(a^{-1} \bullet u). \tag{177c}$$

Definition 18: Let B be a BPFSG of a group G. For any $a \in G$, the bipolar picture fuzzy right coset (BPFRC) is the set

$$Ba(u) = \left\{\sigma^{\pm}_{Ba}(u), \nu^{\pm}_{Ba}(u), \gamma^{\pm}_{Ba}(u)\right\} \tag{178}$$

$$\sigma^{\pm}_{Ba}(u) = \sigma^{\pm}_{B}(u \bullet a^{-1}), \tag{179a}$$
$$\nu^{\pm}_{Ba}(u) = \nu^{\pm}_{B}(u \bullet a^{-1}), \tag{179b}$$
$$\gamma^{\pm}_{Ba}(u) = \gamma^{\pm}_{B}(u \bullet a^{-1}). \tag{179c}$$

Example 6: Let $G = \{1, -1, i, -i\}$ be the multiplicative group and R its BPFSG (indeed its BPFS) defined by

$$\sigma^{+}_{R}(a) = \begin{cases} 0.52, \text{ if } a = 1 \\ 0.45, \text{ if } a = -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma^{-}_{R}(a) = \begin{cases} -0.4, \text{ if } a = 1 \\ -0.3, \text{ if } a = -1 \\ -0.2, \text{ if } a = -i, i \end{cases} \tag{180}$$

$$\gamma^{+}_{R}(a) = \begin{cases} 0.13, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.28, \text{ if } a = -i, i \end{cases} \quad \gamma^{-}_{R}(a) = \begin{cases} -0.1, \text{ if } a = 1 \\ -0.2, \text{ if } a = -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{181}$$

$$\nu^{+}_{R}(a) = \begin{cases} 0.25, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu^{-}_{R}(a) = \begin{cases} -0.5, \text{ if } a = 1 \\ -0.4, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{182}$$

Then, when $a = i$, the BPFLC for σ is

$$\sigma^{+}_{aR}(u) = \begin{cases} 0.52, \text{ if } u = i \\ 0.45, \text{ if } u = -i \\ 0.4, \text{ if } u = -1, 1 \end{cases} \quad \sigma^{-}_{aR}(u) = \begin{cases} -0.4, \text{ if } u = i \\ -0.3, \text{ if } u = -i \\ -0.2, \text{ if } u = -1, 1 \end{cases} \tag{183}$$

Others can be easily computed. Also, the BPFRC can similarly be computed.

Definition 19: Let B be a BPFSG of a group G. For any $a,\ b \in G$, the bipolar picture fuzzy middle coset (BPFMC) is the set

$$aBb(u) = \left\{\sigma^{\pm}_{aBb}(u), \nu^{\pm}_{aBb}(u), \gamma \pm_{aBb}(u)\right\} \tag{184}$$

such that

$$\sigma^{\pm}_{aBb}(u) = \sigma^{\pm}_{B}(a^{-1} \bullet u \bullet b^{-1}), \tag{185a}$$
$$\nu^{\pm}_{aBb}(u) = \nu^{\pm}_{B}(a^{-1} \bullet u \bullet b^{-1}), \tag{185b}$$
$$\gamma^{\pm}_{aBb}(u) = \gamma^{\pm}_{B}(a^{-1} \bullet u \bullet b^{-1}). \tag{185c}$$

Example 7: Let $G = \{1, -1, i, -i\}$ be the multiplicative group and R its BPFSG (indeed its BPFS) defined by

$$\sigma^{+}_{R}(a) = \begin{cases} 0.52, \text{ if } a = 1 \\ 0.45, \text{ if } a = -1 \\ 0.4, \text{ if } a = -i, i \end{cases} \quad \sigma^{-}_{R}(a) = \begin{cases} -0.4, \text{ if } a = 1 \\ -0.3, \text{ if } a = -1 \\ -0.2, \text{ if } a = -i, i \end{cases} \tag{186}$$

$$\gamma^{+}_{R}(a) = \begin{cases} 0.13, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.28, \text{ if } a = -i, i \end{cases} \quad \gamma^{-}_{R}(a) = \begin{cases} -0.1, \text{ if } a = 1 \\ -0.2, \text{ if } a = -1 \\ -0.5, \text{ if } a = -i, i \end{cases} \tag{187}$$

$$\nu^{+}_{R}(a) = \begin{cases} 0.25, \text{ if } a = 1 \\ 0.2, \text{ if } a = -1 \\ 0.1, \text{ if } a = -i, i \end{cases} \quad \nu^{-}_{R}(a) = \begin{cases} -0.5, \text{ if } a = 1 \\ -0.4, \text{ if } a = -1 \\ -0.3, \text{ if } a = -i, i \end{cases} \tag{188}$$

Then, when $a = i$ and $b = -1$, the BPFMC for σ is

$$\sigma^{+}_{aRb}(u) = \begin{cases} 0.52, \text{ if } u = -i \\ 0.45, \text{ if } u = i \\ 0.4, \text{ if } u = -1, 1 \end{cases} \quad \sigma^{-}_{aRb}(u) = \begin{cases} -0.4, \text{ if } u = -i \\ -0.3, \text{ if } u = i \\ -0.2, \text{ if } u = -1, 1 \end{cases} \tag{189}$$

Others can be easily computed.

Definition 20: Let B be a BPFSG of a group G. For any $x \in G$, the normalizer of BPFSG B of G is defined as

$$N(B) = \{a \in G | \sigma_B^+(a^{-1} \bullet x \bullet a) = \sigma_B^+(x), \tag{190}$$
$$\sigma_B^-(a^{-1} \bullet x \bullet a) = \sigma_B^-(x), \tag{191}$$
$$\nu_B^+(a^{-1} \bullet x \bullet a) = \nu_B^+(x), \tag{192}$$
$$\nu_B^-(a^{-1} \bullet x \bullet a) = \nu_B^-(x), \tag{193}$$
$$\gamma_B^+(a^{-1} \bullet x \bullet a) = \gamma_B^+(x), \tag{194}$$
$$\gamma_B^-(a^{-1} \bullet x \bullet a) = \gamma_B^-(x)\} \tag{195}$$

Proposition 13: Let B be a BPFSG of a group G. Then for any $a \in N(B)$ and $v \in G$, $\sigma_B(a \bullet v) = \sigma_B(v \bullet a)$, $\nu_B(a \bullet v) = \nu_B(v \bullet a)$, $\gamma_B(a \bullet v) = \gamma_B(v \bullet a)$, $\sigma_{aBa^{-1}}(a \bullet v) = \sigma_{aBa^{-1}}(v \bullet a)$, $\nu_{aBa^{-1}}(a \bullet v) = \nu_{aBa^{-1}}(v \bullet a)$ and $\gamma_{aBa^{-1}}(a \bullet v) = \gamma_{aBa^{-1}}(v \bullet a)$.
Proof: Let $a \in N(B)$ and $v \in G$. Then

$$\sigma_B^+(a \bullet v) = \sigma_B^+(a^{-1} \bullet a \bullet v \bullet a) = \sigma_B^+(v \bullet a) \Leftrightarrow \sigma_B^+(v) \tag{196}$$
$$= \sigma_B^+(a^{-1} \bullet v \bullet a) \Leftrightarrow \sigma_B^+(v \bullet a) = \sigma_B^+(a^{-1} \bullet v \bullet a \bullet a) \tag{197}$$

and

$$\sigma_{aBa^{-1}}^+(a \bullet v) = \sigma_B^+(a^{-1} \bullet a \bullet v \bullet a) \tag{198}$$
$$= \sigma_B^+(a \bullet v) \tag{199}$$
$$= \sigma_B^+(v \bullet a) \tag{200}$$
$$= \sigma_B^+(a^{-1} \bullet v \bullet a \bullet a) \tag{201}$$
$$= \sigma_{aBa^{-1}}^+(v \bullet a) \tag{202}$$

It can be similarly shown that

$$\sigma_{aBa^{-1}}^-(a \bullet v) = \sigma_{aBa^{-1}}^-(v \bullet a); \tag{203a}$$
$$\nu_{aBa^{-1}}^+(a \bullet v) = \nu_{aBa^{-1}}^+(v \bullet a); \tag{203b}$$
$$\nu_{aBa^{-1}}^-(a \bullet v) = \nu_{aBa^{-1}}^-(v \bullet a); \tag{203c}$$
$$\gamma_{aBa^{-1}}^+(a \bullet v) = \gamma_{aBa^{-1}}^+(v \bullet a); \tag{203d}$$
$$\gamma_{aBa^{-1}}^-(a \bullet v) = \gamma_{aBa^{-1}}^-(v \bullet a). \tag{203e}$$

Remark 5: Always, there exists $e \in N(B)$, where e is the identity in G, since for any $x \in G$, $\sigma_B(x) = \sigma_B(e^{-1} \bullet x \bullet e)$, $\nu_B(x) = \nu_B(e^{-1} \bullet x \bullet e)$ and $\gamma_B(x) = \gamma_B(e^{-1} \bullet x \bullet e)$.
Proposition 14: Let B be a BPFSG of a group G, then for every $a \in G$ the BPFMC aBa^{-1} is a BPFNSG of G.

Proof: Since B is a BPFSG of a group G, then, for any $a \in G$,

$$aBa^{-1}(u) = \left\{\sigma^{\pm}_{aBa^{-1}}(u), \nu^{\pm}_{aBa^{-1}}(u), \gamma^{\pm}_{aBa^{-1}}(u)\right\} \tag{204}$$

is a BPFMC of G for all $u \in G$, where

$$\sigma^{\pm}_{aBa^{-1}}(u) = \sigma^{\pm}_{B}(a^{-1} \bullet u \bullet a), \tag{205a}$$

$$\nu^{\pm}_{aBa^{-1}}(u) = \nu^{\pm}_{B}(a^{-1} \bullet u \bullet a), \tag{205b}$$

$$\gamma^{\pm}_{aBa^{-1}}(u) = \gamma^{\pm}_{B}(a^{-1} \bullet u \bullet a). \tag{205c}$$

Let $u,\ v \in G$. Then,

$$\begin{aligned}
\sigma^{+}_{aBa^{-1}}(u \bullet v^{-1}) &= \sigma^{+}_{B}(a^{-1} \bullet u \bullet v^{-1} \bullet a) && (206)\\
&= \sigma^{+}_{B}(a^{-1} \bullet u \bullet a \bullet a^{-1} \bullet v^{-1} \bullet a) && (207)\\
&= \sigma^{+}_{B}((a^{-1} \bullet u \bullet a) \bullet (a^{-1} \bullet v \bullet a)^{-1}) && (208)\\
&\geq \sigma^{+}_{B}(a^{-1} \bullet u \bullet a) \wedge (\bullet a^{-1} \bullet v \bullet a) && (209)\\
&= \sigma^{+}_{aBa^{-1}}(u) \wedge \sigma^{+}_{aBa^{-1}}(v) && (210)
\end{aligned}$$

$$\begin{aligned}
\sigma^{-}_{aBa^{-1}}(u \bullet v^{-1}) &= \sigma^{-}_{B}(a^{-1} \bullet u \bullet v^{-1} \bullet a) && (211)\\
&= \sigma^{-}_{B}(a^{-1} \bullet u \bullet a \bullet a^{-1} \bullet v^{-1} \bullet a) && (212)\\
&= \sigma^{-}_{B}((a^{-1} \bullet u \bullet a) \bullet (a^{-1} \bullet v \bullet a)^{-1}) && (213)\\
&\leq \sigma^{-}_{B}(a^{-1} \bullet u \bullet a) \vee (\bullet a^{-1} \bullet v \bullet a) && (214)\\
&= \sigma^{-}_{aBa^{-1}}(u) \vee \sigma^{-}_{aBa^{-1}}(v) && (215)
\end{aligned}$$

$$\begin{aligned}
\nu^{+}_{aBa^{-1}}(u \bullet v^{-1}) &= \nu^{+}_{B}(a^{-1} \bullet u \bullet v^{-1} \bullet a) && (216)\\
&= \nu^{+}_{B}(a^{-1} \bullet u \bullet a \bullet a^{-1} \bullet v^{-1} \bullet a) && (217)\\
&= \nu^{+}_{B}((a^{-1} \bullet u \bullet a) \bullet (a^{-1} \bullet v \bullet a)^{-1}) && (218)\\
&\geq \nu^{+}_{B}(a^{-1} \bullet u \bullet a) \wedge (\bullet a^{-1} \bullet v \bullet a) && (219)\\
&= \nu^{+}_{aBa^{-1}}(u) \wedge \nu^{+}_{aBa^{-1}}(v) && (220)
\end{aligned}$$

$$\begin{aligned}
\nu^{-}_{aBa^{-1}}(u \bullet v^{-1}) &= \nu^{-}_{B}(a^{-1} \bullet u \bullet v^{-1} \bullet a) && (221)\\
&= \nu^{-}_{B}(a^{-1} \bullet u \bullet a \bullet a^{-1} \bullet v^{-1} \bullet a) && (222)\\
&= \nu^{-}_{B}((a^{-1} \bullet u \bullet a) \bullet (a^{-1} \bullet v \bullet a)^{-1}) && (223)\\
&\leq \nu^{-}_{B}(a^{-1} \bullet u \bullet a) \vee (\bullet a^{-1} \bullet v \bullet a) && (224)\\
&= \nu^{-}_{aBa^{-1}}(u) \vee \nu^{-}_{aBa^{-1}}(v) && (225)
\end{aligned}$$

Similarly, it can be shown that

$$\gamma^{+}_{aBa^{-1}}(u \bullet v^{-1}) \leq \gamma^{+}_{aBa^{-1}}(u) \vee \gamma^{+}_{aBa^{-1}}(v) \tag{226}$$

and

$$\gamma^{-}_{aBa^{-1}}(u \bullet v^{-1}) \geq \gamma^{-}_{aBa^{-1}}(u) \wedge \gamma^{-}_{aBa^{-1}}(v). \tag{227}$$

Hence, aBa^{-1} is a BPFSG of G.
Next, to show that BPFMC is a BPFNSG of G.
Let $a,\ v \in G$.
Then, by Proposition 13,

$$\sigma^{+}_{aBa^{-1}}(a \bullet v) = \sigma^{+}_{B}(a^{-1} \bullet (a \bullet v) \bullet a) \tag{228}$$

$$= \sigma^{+}_{B}(a \bullet v) = \sigma^{+}_{B}(v \bullet a) \tag{229}$$

$$\Leftrightarrow a \in N(B) \Leftrightarrow \sigma^{+}_{B}(a \bullet (v \bullet a)) \tag{230}$$

$$= \sigma^{+}_{B}((v \bullet a) \bullet a) \Leftrightarrow \sigma^{+}_{B}(v \bullet a) \tag{231}$$

$$= \sigma^{+}_{B}(a^{-1} \bullet (v \bullet a) \bullet a) \tag{232}$$

$$= \sigma^{+}_{aBa^{-1}}(v \bullet a) \tag{233}$$

Similarly,

$$\sigma^{+}_{aBa^{-1}}(a \bullet v) = \sigma^{+}_{aBa^{-1}}(v \bullet a) \tag{234a}$$

$$\sigma^{-}_{aBa^{-1}}(a \bullet v) = \sigma^{-}_{aBa^{-1}}(v \bullet a) \tag{234b}$$

$$\nu^{+}_{aBa^{-1}}(a \bullet v) = \nu^{+}_{aBa^{-1}}(v \bullet a) \tag{234c}$$

$$\nu^{-}_{aBa^{-1}}(a \bullet v) = \nu^{-}_{aBa^{-1}}(v \bullet a) \tag{234d}$$

$$\gamma^{+}_{aBa^{-1}}(a \bullet v) = \gamma^{+}_{aBa^{-1}}(v \bullet a) \tag{234e}$$

$$\gamma^{-}_{aBa^{-1}}(a \bullet v) = \gamma^{-}_{aBa^{-1}}(v \bullet a). \tag{234f}$$

Hence, the BPFMC aBa^{-1} is a BPFNSG of G.

5. CONCLUDING REMARKS

This paper has introduced the concepts of bipolar picture fuzzy subgroup of a group, bipolar picture fuzzy cosets. This bipolar picture fuzzy subgroup of a group is an extension of bipolar picture fuzzy sets and also a generalisation of both bipolar fuzzy subgroup and bipolar intuitionistic fuzzy subgroup. Several properties of bipolar picture fuzzy subgroup were established.

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