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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS OF AIZERMANN VECTOR DIFFERENTIAL EQUATIONS

A. A. ADEYANJU[1], M. O. OMEIKE[2], J. O. ADENIRAN[3] AND B. S. BADMUS[4]

ABSTRACT. The objective of this paper is to examine certain sufficient conditions for the uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of all solutions to a certain Aizermann vector differential equation. By constructing an appropriate complete Lyapunov function, we provide sufficient conditions that guarantee the qualitative properties mentioned above. The results of this paper are solutions to the open problems contained in Ezeilo [18].

1. INTRODUCTION

In this paper, we consider the following systems of first order Aizermann differential equations

$$\dot{X} = F(X) + H(Y) + P_1(t,X,Y),\ \dot{Y} = CX + DY + P_2(t,X,Y), \tag{1}$$

where $X,\ Y\ \in \mathbb{R}^n$; C, and D are real $n \times n$ symmetric constant matrices; $F, H : \mathbb{R}^n \to \mathbb{R}^n$ are one time continuously differentiable functions(C^1) satisfying $F(0) = G(0) = 0$, and $P_1, P_2 : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. To guarantee existence and uniqueness of solutions of (1), the nonlinear terms in (1) are assumed to be continuous and satisfy Lipschitz continuity condition.

Qualitative theory of differential equations which began in late 19th century by the works of Poincaré [37] and Lyapunov [29] has since been receiving a great attention from researchers and experts across the world on the subject till date. A very useful tool in the study of qualitative properties of solutions of differential equations is a scalar function known as Lyapunov function named after A. M. Lyapunov who introduced the two popular methods known today as the first and second methods of Lyapunov. This function, under certain conditions helps to deduce information on the behaviour of solutions of differential equations without necessarily solving the equations themselves (See, [49], [50]). The second method of Lyapunov has been employed by many authors to study stability, boundedness, convergence, periodicity etc. of solutions to many differential equations, see the works of Adeyanju([2], [3], [4]), Adeyanju and Tunc ([5], [6]) Adeyanju et. al [7], Adeyanju and Adams[8], Burton[13], Cartwright[14], Erugin[15], Ezeilo[17], Graef[23], Huiqing[24], Jiang[25], Loud[28], Malkin[30], Mufti[31], Omeike([33], [34]), Qian([38]-[40]), Sugie[42], Tejumola[43], Tunc([44], [45]), and Yoshizawa[48].

In 1949, Aizerman (Aizerman [10], Parks [35]) proposed the following problem. Let there be given a system of linear differential equations

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + ax_1,\ \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j,\ (i = 2,3,...,n). \tag{2}$$

Suppose that for the given constants a_{ij} $(i, j = 1,2,...,n)$ and for an arbitrary value of a from the interval $\alpha < a < \beta$, all the roots of the characteristics equation of system (2) have negative real parts. Let ax_1

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be replaced by $f(x_1)$ in (2). We then have:

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + f(x_1),\ \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j,\ (i = 2,3,...,n). \tag{3}$$

Aizermann's problem (Boikov [12]) is now as follows. If it is known that the trivial solution of the linear system (2) is asymptotically stable for all a satisfying the condition $\alpha < a < \beta$, will the trivial solution of the non-linear system (3) be stable in the large if the following condition is satisfied

$$\alpha x_1^2 < x_1 f(x_1) < \beta x_1^2, \text{ for } x_1 \neq 0. \tag{4}$$

This problem has inspired many researches. In fact, it has been shown that the condition (4) is not sufficient for stability in second-order (Krasovskii [27]) and third-order (Pliss [36]) systems. In other words, it is required to find out whether the trivial solution $x_1 = x_2 = ... = x_n = 0$ of the system (3) is asymptotically stable in the large or not, for an arbitrary choice of a continuous function $f(x_1)$, which reduces to zero for $x_1 = 0$ and which satisfies the inequality (4).

Particularly, Erugin [16] examined the stability property of the trivial solution to the one-dimensional Aizerman equation

$$\dot{x} = \psi(x) + by,\ \dot{y} = cx + dy,$$

where b, c and d are constants and $\psi(x)$ is a continuous scalar function. Mufti [32] considered the following equation

$$\dot{x} = ay + xf(y),\ \dot{y} = bx + yg(x),$$

where a, b are constants. Much later, Ezeilo [18] proved some results on the stability of the trivial solution to the following Aizerman vector differential equations

$$\dot{X} = F(X) + BY,\ \dot{Y} = G(X) + DY, \tag{5}$$

and

$$\dot{X} = AY - F(Y)X,\ \dot{Y} = -BX - G(X)Y, \tag{6}$$

where $X,\ Y \in \mathbb{R}^n$, A, B and D are real $n \times n$ constant matrices and $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

Furthermore, in a quite recent paper, Adeyanju et.al. [1] proved some interesting results on the stability and boundedness of solutions to the following vector form of Aizermann differential equation

$$\dot{X} = F(X) + BY + P_1(t,X,Y),\ \dot{Y} = G(X) + DY + P_2(t,X,Y), \tag{7}$$

where $X,\ Y\ \in \mathbb{R}^n$, B, and D are real $n \times n$ symmetric constant matrices, $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are one time continuously differentiable functions(C^1) satisfying $F(0) = G(0) = 0$.

Inspired by the works of Ezeilo [18], Krasovskii ([26], [27]), Adeyanju et.al. [1] and other listed references, we are going to use the direct method of Lyapunov to prove some results on the stability and boundedness of solutions of (1). The stability and boundedness of solutions to (1) and other similar Aizermann vector differential equations, until now remain an open problem due to difficulty in constructing a suitable Lyapunov function [18].

Remark 1.1.
(i) We introduced the terms $P_1(t,X,Y)$ and $P_2(t,X,Y)$ appearing in (1) so as to study other qualitative properties of solutions of (1) aside stability.
(ii) The qualitative properties of (1) have not been discussed in literature by any author to the best of our understanding.

Consider a system of differential equations (Yoshizawa [47])

$$\dot{X} = f(t,X), \tag{8}$$

where X is an n-vector and $f(t,X)$ is an n- vector function which is defined in a region $\Omega \subset I \times \mathbb{R}^n$ and continuous in (t_0, X_0) so that for each (t_0, X_0) there is a solution $X(t; t_0, X_0)$ satisfying

$$X(t; t_0, X_0) = X, \tag{9}$$

and

$$X(t_0; t_0, X_0) = X_0. \tag{10}$$

Let f be Lipschitz and continuous so as to ensure the existence of a unique solution of equation (8). Then, we can give the following definitions and theorems about the solutions of equation (8).

Definition 1.2. Stability and Asymptotic Stability (Yoshizawa [47]).
A solution $\phi(t)$ of (1) defined for $t \geq 0$, is said to be Lyapunov stable if given an $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\varphi(t)$ of (8) with:

$$\|\varphi(0) - \phi(0)\| < \delta \tag{11}$$

satisfies

$$\|\varphi(t) - \phi(t)\| < \varepsilon \tag{12}$$

for all $t \geq 0$, where $\|.\|$ stands for norm.

If in addition to the definition of stability above, we have:

$$\|\varphi(t) - \phi(t)\| \to 0 \text{ as } t \to \infty, \tag{13}$$

then we say the solution $\phi(t)$ is asymptotically stable.

Definition 1.3. Boundedness (Yoshizawa [47])
A solution $\phi(t)$ of (8) is said to be bounded if there exist a $\beta > 0$ and a constant $M > 0$ such that $\|\phi(t, t_0, x_0)\| < M$ whenever $\|x_0\| < \beta$, $t \geq t_0$.

We shall consider the differential system (8) under the assumption that $f(t,X)$ is continuous on $0 \leq t < \infty$, $\|X\| < H$, and $f(t,0) \equiv 0$.

Theorem 1.4. (Yoshizawa [47])
Suppose that there exists a Lyapunov function $V(t,X)$ defined on $0 \leq t < \infty$, $\|X\| < H$ which satisfies the following conditions:

(i) $V(t,0) = 0$
(ii) $a(\|X\|) \leq V(t,X) \leq b(\|X\|)$,
(iii) $\dot{V}_{(8)}(t,X) \leq -c(\| X \|)$,

where $a(r), b(r)$ and $c(r)$ are continuous-increasing positive definite function(CIP). Then, the zero solution of (8) is uniform-asymptotically stable.

Theorem 1.5. (Yoshizawa [47])
Suppose that there exists a Lyapunov function $V(t,X)$ defined on $0 \leq t < \infty$, $\|X\| \geq D$, (where D may be large) which satisfies:

(i) *$a(\|X\|) \leq V(t,X) \leq b(\|X\|)$, where $a(r)$ and $b(r)$ are continuous, monotone increasing functions and $a(r) \to \infty$ as $r \to \infty$,*
(ii) *$\dot{V}(t,X) \leq -c(\|X\|)$, where $c(r)$ is positive and continuous.*

Then, the solutions of equation (8) are uniformly ultimately bounded.

Theorem 1.6. (LaSalle's Invariance Principle)(Tunc and Mohammed [44])
If V is a Lyapunov function on a set G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geq 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in G^* : V(\psi) = 0\}$, where G^* is the closure of set G and ω denotes the omega limit set of a solution.

2. Preliminary Results

Here, we state some known results that will be helpful in the proofs of our main results later.

Lemma 2.1. ([19], [22], [33], [45])
Let A be a real symmetric $n \times n$-constant matrix and

$$\delta_a \leq \lambda_i(A) \leq \Delta_a, \ (i = 1,2,...,n),$$

where δ_a and Δ_a are constants representing the least and greatest eigenvalues of matrix A respectively. Then,

$$\delta_a \langle X, X\rangle \leq \langle AX, X\rangle \leq \Delta_a \langle X, X\rangle.$$

Lemma 2.2. ([18])
Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and suppose that $H(0) = 0$.

(i) *Then for any $X \in \mathbb{R}^n$,*

$$H(X) = \int_0^1 J_h(sX)X ds,$$

where $J_h(X)$ is the Jacobian matrix of $H(X)$;

(ii) *Let $J_h(X)$ be symmetric and commutes with a certain real symmetric $n \times n$ matrix E. Then*

$$\frac{d}{dt}\int_0^1 \langle EH(sX), X\rangle ds = \langle EH(X), \dot{X}\rangle,$$

for any real differentiable vector $X = X(t) \in \mathbb{R}^n$.

Lemma 2.3. ([9], [20], [21]) *Let A, B be any two real symmetric positive definite $n \times n$ matrices. Then,*

(i) *the eigenvalues $\lambda_i(AB)$, $(i = 1,2,\ldots,n)$, of the product matrix AB are real and satisfy*

$$\min_{1\leq j,k\leq n} \lambda_j(A)\lambda_k(B) \leq \lambda_i(AB) \leq \max_{1\leq j,k\leq n} \lambda_j(A)\lambda_k(B);$$

(ii) *the eigenvalues $\lambda_i(A+B)$, $(i = 1,2,\ldots,n)$, of the sum of matrices A and B are real and satisfy*

$$\{\min_{1\leq j\leq n} \lambda_j(A) + \min_{1\leq k\leq n} \lambda_k(B)\} \leq \lambda_i(A+B) \leq \{\max_{1\leq j\leq n} \lambda_j(A) + \max_{1\leq k\leq n} \lambda_k(B)\}.$$

3. Main Results

In this section, we state and prove our main results regarding system (1). The following estimates defined for the matrices in the brackets below, will be used later to establish the proofs of main results.

Let, $\delta_h, \delta_c, \delta_1, \delta_2, \delta_1, \Delta_h, \Delta_c$ and Δ_2 be some constants. Then, we define the following for $(i = 1,2,\ldots,n)$:

(i) $0 < \delta_h \leq \lambda_i(-J_h(Y)) \leq \Delta_h, 0 < \delta_c \leq \lambda_i(C) \leq \Delta_c$;
(ii) $-\Delta_1 \leq \lambda_i(-J_hD) \leq -\delta_1 < 0, -\Delta_2 \leq \lambda_i(CJ_f) \leq -\delta_2 < 0$.

Theorem 3.1. *Suppose $J_f(X)$, $J_h(Y)$ denote the Jacobian matrices $\frac{\partial f_i}{\partial x_i}$, $\frac{\partial h_i}{\partial y_i}$ of $F(X)$ and $H(Y)$ respectively and $F(0) = 0, H(0) = 0$, $P_1(t,X,Y) = 0, P_2(t,X,Y) = 0$. Further,*

(i) *the matrices $D, J_h(Y)$, $J_f(X)$ are all symmetric and negative definite, while C is a symmetric and positive definite matrix,*
(ii) *the matrices C, D, J_h and J_f commutes with each other,*
(iii) *the matrix $\{DJ_f(X) - CJ_h(X)\}$ is strictly positive definite.*

Then the trivial solution of system (1) *is uniformly-asymptotically stable and satisfies*

$$\|X(t)\| \to 0, \ \|Y(t)\| \to 0, \ as\ t \to \infty. \tag{14}$$

Theorem 3.2. *Under the assumptions (i)-(iii) of Theorem 3.1, if*

(iv) $\|P_1(t,X,Y)\| \leq \phi(t)$, $\|P_2(t,X,Y)\| \leq \theta(t)$ *for all* $t \geq 0$, $\max\theta(t) < \infty$, $\max\phi(t) < \infty$ *and* $\theta(t), \phi(t) \in L^1(0,\infty)$, *where $L^1(0,\infty)$ is the space of integrable Lebesgue functions.*

Then solutions of system (1) *are bounded.*

Theorem 3.3. *Under the assumptions (i)-(iii) of Theorem 3.1, if*

(iv) $\|P_1(t,X,Y)\| \le \phi(t)\{1+\|X\|\}$, $\|P_2(t,X,Y)\| \le \theta(t)\{1+\|Y\|\}$ *for all* $t \ge 0$, $\max\theta(t) < \infty$, $\max\phi(t) < \infty$ *and* $\theta(t)$, $\phi(t) \in L^1(0,\infty)$, *where* $L^1(0,\infty)$ *is the space of integrable Lebesgue functions.*

Then, any solution $(X(t),Y(t))$ *of system* (1) *with the initial condition*

$$X(0) = X_0, Y(0) = Y_0, \tag{15}$$

satisfies

$$\|X(t)\| \le A_1,\ \|Y(t)\| \le A_1 \tag{16}$$

for all $t \ge 0$ *where* $A_1 > 0$ *depends on* $C,\ D,\ \theta(t), t_0,\ X_0,\ Y_0,$ *and* $P_i(t,X,Y),(i = 1.2)$.

The main tool in the proof of these theorems is the Lypunov function defines as

$$2V(X,Y) = \langle X,CX\rangle - 2\int_0^1 \langle H(sY),Y\rangle ds. \tag{17}$$

The next lemma is useful in providing proofs to the main results.

Lemma 3.4. *Suppose that, under the assumptions of Theorem 3.1 there exist constants* A_2 *and* A_3 *both positive such that the function* V *defined by equation* (17)*, satisfies*

$$A_2\{\|X(t)\|^2 + \|Y(t)\|^2\} \le V(X,Y) \le A_3\{\|X\|^2 + \|Y\|^2\}, \tag{18}$$

and

$$V(X(t),Y(t)) \to +\infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{19}$$

Furthermore, there exists a positive constant A_4 *such that for any solution* $(X(t),Y(t))$ *of* (1) *we have*

$$\dot{V}(X,Y) \le -A_4\{\|X(t)\|^2 + \|Y(t)\|^2\}, \tag{20}$$

for all $t \ge 0$ *and* $X,Y \in \mathbb{R}^n$.

Proof. It is clear that for $X(t) = Y(t) = 0,\ t \ge 0,\ V(X,Y) = 0$. Also, from the scalar function defined in equation (17) above, we have

$$\begin{aligned} 2V(X,Y) =& \langle X,CX\rangle - 2\int_0^1 \langle H(sY),Y\rangle ds \\ =& \langle X,CX\rangle - 2\int_0^1\int_0^1 \langle J_h(s_1s_2Y)Y,Y\rangle s_1 ds_1 ds_2. \end{aligned}$$

By the hypotheses of Theorem 3.1 and Lemma 2.1, we have,

$$-\langle J_h(s_1s_2Y)Y,Y\rangle \ge \delta_h\|Y\|^2,$$

and

$$\langle X,CX\rangle \ge \delta_c\|X\|^2.$$

Thus,

$$\begin{aligned} \int_0^1\int_0^1 \langle J_h(s_1s_2Y)Y,Y\rangle s_1 ds_1 ds_2 \ge& \delta_h\|Y\|^2 \int_0^1\int_0^1 s_1 ds_1 ds_2 \\ =& \frac{1}{2}\delta_h\|Y\|^2. \end{aligned}$$

Hence,

$$2V(X,Y) \ge \frac{1}{2}\delta_h\|Y\|^2 + \delta_c\|X\|^2.$$

There exists a constant $A_2 = \min\{\frac{1}{2}\delta_h, \delta_c\}$, such that

$$2V(X,Y) \ge A_2(\|X\|^2 + \|Y\|^2), \tag{21}$$

for all $t \ge 0$ and $X,Y \in \mathbb{R}^n$. It then follows from (21) that $V(X,Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(X,Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \ne 0$, which now implies that

$$V(X,Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{22}$$

Similarly, by the hypotheses of Theorem 3.1 and Lemma 2.1, we obtain

$$-\langle J_h(s_1s_2Y)Y,Y\rangle \leq \Delta_h\|Y\|^2,$$

and

$$\langle X,CX\rangle \leq \Delta_c\|X\|^2.$$

Therefore,

$$\begin{aligned}-\int_0^1\int_0^1\langle J_h(s_1s_2Y)Y,Y\rangle s_1ds_1ds_2 \leq &\Delta_h\|Y\|^2\int_0^1\int_0^1 s_1ds_1ds_2\\ =&\frac{1}{2}\Delta_h\|Y\|^2.\end{aligned}$$

Thus,

$$2V(X,Y) \leq \frac{1}{2}\Delta_h\|Y\|^2+\Delta_c\|X\|^2.$$

Then, we can always find a constant $A_3=\max\{\frac{1}{2}\Delta_h,\Delta_c\}$, such that

$$2V(X,Y)\leq A_3(\|X\|^2+\|Y\|^2), \tag{23}$$

for all $t\geq 0$ and $X,Y\in\mathbb{R}^n$. Hence,

$$A_2\{\|X\|^2+\|Y\|^2\}\leq 2V(X,Y)\leq A_3\{\|X\|^2+\|Y\|^2\}.$$

In what follows, we obtain the derivative of V with respect to t along the solution path of the system (1) such that it satisfies

$$\dot{V}|_{(1)}\equiv\frac{d}{dt}V(X,Y)|_{(1)}\leq -A_5 \tag{24}$$

provided that $\|X\|^2+\|Y\|^2\leq A_6$, both A_5 and A_6 are some positive constants. The derivative of function V in (17) is

$$\begin{aligned}\dot{V}(X,Y)=&-\langle H(Y),DY\rangle+\langle CX,F(X)\rangle\\ =&-\int_0^1\langle J_h(s_1Y)Y,DY\rangle ds_1+\int_0^1\langle CX,J_f(s_1X)X\rangle ds_1.\end{aligned}$$

By the hypotheses of Theorem 3.1 and Lemma 2.3, we have

$$\dot{V}(X,Y)\leq -\delta_1\|X\|^2-\delta_2\|Y\|^2. \tag{25}$$

Thus, there exists a constant $A_4=\min\{\delta_1,\delta_2\}>0$ such that

$$\dot{V}(X,Y)\leq -A_4\{\|X\|^2+\|Y\|^2\}, \tag{26}$$

for all $X,Y\in\mathbb{R}^n$. This completes the proof of Lemma 3.4. □

Proof. Theorem 3.1
From inequalities (21), (23) and (26) in the proof of Lemma 3.4, we therefore conclude that, the trivial/zero solution of system (1) is uniformly stable.

Furthermore, consider the set W defined by

$$W=\{(X,Y):\dot{V}(X,Y)=0\}.$$

By using LaSalle's invariance principle, we observe that $(X,Y)\in W$ implies that $X=Y=0$. Hence, this shows that the largest invariant set contained in W is $(0,0)\in W$. Therefore, the zero solution of system (1) is uniformly-asymptotically stable since all the conditions of Theorem 1.4 are being satisfied. This completes the proof of Theorem 3.1. □

Proof. Theorem 3.2
The inequalities (21) and (23) obtained earlier are still much valid here. Hence, we proceed to obtain the derivative of the function V defined in (17) when $P_1(t,X,Y) \neq 0$ and $P_2(t,X,Y) \neq 0$ as follows.

$$\begin{aligned} \dot{V}(X,Y) &= -\langle H(Y), DY\rangle + \langle CX, F(X)\rangle + \langle CX, P_1(t,X,Y)\rangle - \langle H(Y), P_2(t,X,Y)\rangle \\ &= -\int_0^1 \langle J_h(s_1Y)Y, DY\rangle ds_1 + \int_0^1 \langle CX, J_f(s_1X)X\rangle ds_1 + \langle CX, P_1(t,X,Y)\rangle \\ &\quad - \langle H(Y), P_2(t,X,Y)\rangle. \end{aligned} \tag{27}$$

By the hypotheses of Theorem 3.2, Lemma 2.3 and Lemma 3.4, we have

$$\dot{V}(X,Y) \leq -\delta_1\|X\|^2 - \delta_2\|Y\|^2 + \langle CX, P_1(t,X,Y)\rangle - \langle H(Y), P_2(t,X,Y)\rangle.$$

But,

$$\begin{aligned} -\langle H(Y), P_2(t;X,Y)\rangle &\leq |\langle H(Y), P_2(t,X,Y)\rangle| \\ &\leq \int_0^1 |\langle J_h(s_1Y)Y, P_2(t,X,Y)\rangle| ds_1 \\ &\leq A_7\|Y\|\|P_2(t,X,Y)\| \end{aligned}$$

where $A_7 = \Delta_h$ and has been defined earlier.
Also,

$$\begin{aligned} \langle CX, P_1(t,X,Y)\rangle &\leq |\langle CX, P_1(t,X,Y)\rangle| \\ &\leq \Delta_c\|X\|\|P_1(t,X.Y)\|. \end{aligned}$$

Now, on using the inequalities

$$\|Y\| \leq 1 + \|Y\|^2, \ \|X\| \leq 1 + \|X\|^2$$

and the hypothesis (vi) of Theorem 3.2, our estimate for $\dot{V}$ becomes

$$\begin{aligned} \dot{V} &\leq A_7\{1+\|Y\|^2\}\|P_2(t,X,Y)\| + \Delta_c\{1+\|X\|^2\}\|P_1(t,X.Y)\| \\ &\leq A_7\{1+\|Y\|^2\}\theta(t) + \Delta_c\{1+\|X\|^2\}\phi(t) \\ &\leq A_7\theta(t) + \Delta_c\phi(t) + A_7\theta(t)\|Y\|^2 + \Delta_c\phi(t)\|X\|^2. \end{aligned}$$

The following facts are obvious from inequality (21),

$$\|Y\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2A_2^{-1}V(X,Y)$$

and

$$\|X\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2A_2^{-1}V(X,Y).$$

Therefore,

$$\dot{V} \leq A_7\theta(t) + \Delta_c\phi(t) + 2A_2^{-1}\{A_7\theta(t) + \Delta_c\phi(t)\}V(X,Y) \tag{28}$$

for all $t \geq 0, X, Y \in \mathbb{R}^n$. On setting $\theta_5(t) = A_7\theta(t) + \Delta_c\phi(t)$ in (28), we obtain

$$\dot{V} \leq \theta^*(t) + 2A_2^{-1}\theta_5(t)V(X,Y). \tag{29}$$

The integration of both sides of (29) between 0 to t, $(t > 0)$, leads to

$$V(t) \leq V(0) + \int_0^t \theta_5(s)ds + 2A_2^{-1}\int_0^t \theta_5(s)V(s)ds.$$

Let

$$W_1 = V(0) + \int_0^\infty \theta_5(s)ds \text{ and } W_2 = 2A_2^{-1}.$$

Then,

$$V(t) \leq W_1 + W_2\int_0^\infty V(s)\theta_5(s)ds.$$

By applying Gronwall-Bellman inequality [41], we have

$$V(t) \leq W_1 \exp(W_2 \int_0^\infty \theta_5(s)ds) \leq A_8,$$

where A_8 is a positive constant. By the estimates (21), (23) and the assumptions on $\theta(t)$, we can conclude that all solutions of (1) are uniformly bounded according to Theorem 1.5. Hence, the proof of Theorem 3.2 is complete. □

Proof. Theorem 3.3
The proof of this theorem is similar to that of Theorem 3.2. But in this case, $\|P_1(t,X,Y)\| \leq \phi(t)\{1+\|X\|\}$ and $\|P_2(t,X,Y)\| \leq \theta(t)\{1+\|Y\|\}$. Already, we know from the proof of Theorem 3.2 that

$$\dot{V} \leq A_7\{\|Y\|\}\|P_2(t,X,Y)\| + \Delta_c\{\|X\|\}\|P_1(t,X.Y)\|.$$

On using the hypotheses of Theorem 3.3, we obtain

$$\begin{aligned}\dot{V} \leq& A_7\theta(t)\|Y\|\{1+\|Y\|\} + \Delta_c\phi(t)\|X\|\{1+\|X\|\}\\ \leq& A_7\theta(t)\{\|Y\|+\|Y\|^2\} + \Delta_c\phi(t)\{\|X\|+\|X\|^2\}.\end{aligned}$$

By using the following facts,

$$\|Y\| \leq 1+\|Y\|^2 \text{ and } \|X\| \leq 1+\|X\|^2,$$

we have,

$$\dot{V} \leq A_7\theta(t)\{1+2\|Y\|^2\} + \Delta_c\phi(t)\{1+2\|X\|^2\}$$

$$\dot{V} \leq A_7\theta(t) + \Delta_c\phi(t) + 2A_7\theta(t)\|Y\|^2 + 2\Delta_c\phi(t)\|X\|^2. \tag{30}$$

Now, from the inequality (21), we have the following facts

$$\|X\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2A_2^{-1}V(X,Y) \tag{31}$$

and

$$\|Y\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2A_2^{-1}V(X,Y). \tag{32}$$

Thus, on applying these facts in (30) above and setting $\theta_5 = A_7\theta(t) + \Delta_c\phi(t)$, we obtain

$$\dot{V} \leq \theta_5(t) + 2A_2^{-1}\theta_5(t)V(X,Y), \tag{33}$$

for all $t \geq 0, X, Y$. The integration of both sides of (33) between 0 to t, $(t > 0)$, gives

$$V(t) \leq V(X(0),Y(0)) + \int_0^t \theta_5(s)ds + 2A_2^{-1}\int_0^t \theta_5(s)(s)V(s)ds.$$

Let

$$W_3 = V(X(0),Y(0)) + \int_0^\infty \theta_5(s)ds \text{ and } W_4 = 2A_2^{-1}.$$

Then,

$$V(t) \leq W_3 + W_4\int_0^\infty V(s)\theta_5(s)ds.$$

By applying Gronwall-Bellman inequality [41], we have

$$V(t) \leq W_3 \exp(W_4 \int_0^\infty \theta_5(s)ds) \leq A_9 \tag{34}$$

where $A_9 > 0$ is a constant. On using (21) in the inequality (34), we obtain

$$\|X\|^2 + \|Y\|^2 \leq 2A_9A_2^{-1} = A_1$$

and this implies

$$\|X\|^2 \leq A_1 \text{ and } \|Y\|^2 \leq A_1.$$

This completes the proof of Theorem 3.3. □

4. Ultimate Boundedness of Solutions

Theorem 4.1. *Under the assumptions of Theorem 3.2 or Theorem 3.3, all the solutions of system* (1) *are uniformly-ultimately bounded.*

Proof. From the proof of Theorem 3.2 we have that the derivative $\dot{V}$ of the function V defined in (17), satisfied

$$\begin{aligned}\dot{V} \leq &- \delta_1\|X\|^2 - \delta_2\|Y\|^2 + \langle CX, P_1(t,X,Y)\rangle - \langle H(Y), P_2(t,X,Y)\rangle \\ \leq &- \delta_1\|X\|^2 - \delta_2\|Y\|^2 + \Delta_c\|X\|\|P_1(t,X,Y)\| + \Delta_h\|Y\|\|P_2(t,X,Y)\|\end{aligned}$$

for some positive constants Δ_c and Δ_h. From the hypotheses of the theorem, we obtain

$$\begin{aligned}\dot{V} \leq &- \delta_1\|X\|^2 - \delta_2\|Y\|^2 + \Delta_c\phi(t)\|X\|\{1+\|X\|\} + \Delta_h\theta(t)\|Y\|\{1+\|Y\|\} \\ \leq &- \delta_1\|X\|^2 - \delta_2\|Y\|^2 + \Delta_c\phi(t)\{\|X\|+\|X\|^2\} + \Delta_h\theta(t)\{\|Y\|+\|Y\|^2\}.\end{aligned}$$

Now, suppose $A_{10} = \max\{\Delta_c, \Delta_h\}$ and $0 \leq \delta_3 = \max\{\theta(t), \phi(t)\}$, we have

$$\dot{V} \leq -\delta_3\|X\|^2 - \delta_2\|Y\|^2 + \delta_3 A_{10}\{\|X\|+\|Y\|\} + A_{10}\delta_3\{\|X\|^2+\|Y\|^2\}.$$

Thus, from the conclusion of Lemma 3.4, we obtain

$$\dot{V} \leq -A_4\{\|X\|^2+\|Y\|^2\} + \delta_3 A_{10}\{\|X\|+\|Y\|\} + A_{10}\delta_3\{\|X\|^2+\|Y\|^2\}.$$

On using the fact that

$$\{\|X\|+\|Y\|\} \leq 2^{\frac{1}{2}}\{\|X\|^2+\|Y\|^2\}^{\frac{1}{2}},$$

we have

$$\dot{V} \leq -\{A_4 - A_{10}\delta_3\}\{\|X\|^2+\|Y\|^2\} + 2^{\frac{1}{2}}A_{10}\delta_3\{\|X\|^2+\|Y\|^2\}^{\frac{1}{2}}. \tag{35}$$

By letting $\delta_4 = \frac{1}{2}(A_4 - A_{10}\delta_3)$, $\delta_3 < A_4 A_{10}^{-1}$ and $\delta_5 = 2^{\frac{1}{2}}\alpha_1 A_{10}$, we have

$$\dot{V} \leq -2\delta_4\{\|X\|^2+\|Y\|^2\} + \delta_5\{\|X\|^2+\|Y\|^2\}^{\frac{1}{2}}. \tag{36}$$

If we choose $(\|X\|^2+\|Y\|^2)^{\frac{1}{2}} \geq \delta_6 = \delta_5\delta_4^{-1}$, then the inequality (36) above implies that

$$\dot{V} \leq -\delta_4\{\|X\|^2+\|Y\|^2\}. \tag{37}$$

The remaining part of the proof follows exactly the Yoshizawa techniques found in Yoshizawa [48]. □

5. Conclusion

In this paper, we have studied some conditions for the uniform-asymptotic stability of the trivial solution and uniform-ultimate boundedness of all solutions to a particular Aizermann vector differential equation by means of Lyapunov direct method. By the results of this paper, the age-long Aizermann problems have been brought to limelight for further study be interested researchers.

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[1]Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria
email: adeyanjuaa@funaab.edu.ng

[2]Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria
email: omeikemo@funaab.edu.ng

[3]Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria
email: adeniranoj@funaab.edu.ng

[4]Department of Physics, Federal University of Agriculture, Abeokuta, Nigeria
email: badmusbs@funaab.edu.ng