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ON THE CONVERGENCE BEHAVIOUR OF SOLUTIONS OF CERTAIN SYSTEM OF SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Convergence criteria for the solutions of certain system of two nonlinear delay differential equations with continuous deviating argument $\varrho(t)$ using a suitable Lyapunov-Krasovskii's functional are established in this study. The new result attained extends and updates some results mentioned in the literature. A numerical illustration is given to show the validity of the result as well geometric analysis to describe the behavior of solutions of the system.

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1. INTRODUCTION

We consider here system of nonlinear differential equations of the form,

$$\begin{aligned} \dot{X} &= H(Y) \\ \dot{Y} &= -\Phi(X,Y)Y - \Psi(X(t-\varrho(t)) + P(t,X,Y), \end{aligned} \tag{1}$$

where $X, Y \in \mathbb{R}^n$ are the unknown functions, $H, \Psi : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and H, Φ, Ψ, P are continuous in their respective arguments. $0 \le \varrho(t) \le \xi$, $(\xi > 0)$, $\varrho'(t) \le \upsilon$, $0 \le \upsilon \le 1$ and ξ value will be fixed later. Moreover, $\varrho(t)$ is once continuously differentiable on $\mathbb{R}^+$ and we also assume that

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the solutions of (1) exist and are unique. Equation (1) is the vector version for the system

$$\dot{x}_i = h_i\left(y_1, y_2, \ldots, y_n\right)$$
$$\dot{y}_i = -\sum_{k=1}^{n} \Phi_{ik} y_k - \Psi_i(x_1(t-\varrho(t)), (x_2(t-\varrho(t))..., (x_n(t-\varrho(t)))$$
$$+ p_i\left(t; x_1, x_2, \ldots, x_n; y_1, \ldots, y_n\right), \qquad (i = 1, 2, \ldots, n)$$

in which Φ_{ik} are functions of x_i, y_i. If $i = k = 1$ and $\varrho(t) = 0$, the system reduces to the one studied by Tejumola [14] for boundedness of solutions.

A lot of results have been obtained on the study of properties of solutions of ordinary differential equations. The essence of these properties of solutions is to characterize and describe the nature of solutions of ordinary differential equations as well as the state variables defining models of natural phenomena and dynamical systems. Moreover, properties of solution of delay differential equations are very vital in specifying the behaviour of solutions of fairly complicated nonlinear systems arising from some fields of science and technology such as after effect, nonlinear oscillation, biological systems and equations with deviating arguments. (See, Cronin [6] and Rauch [13]). Results abound for convergence of solutions for various second and third order certain nonlinear scalar or vector differential equations. (See for example, [9], [10], [11], [15], [16], [17], [18], [19], [20] and the references cited therein). But with respect to system of two nonlinear vector differential equations, only very few results exist in the literature for the stability and boundedness of solutions. (See for example, [12] and [8]). Scarce still, is the convergence of solutions of certain system (1) with variable delay. Though it goes without saying that studies on delay differential equation are ubiquitous but system of two nonlinear delay differential equations are less visible and rarely scarce. More specifically, system of the form (1) has not been discussed in the relevant literature. This new system will be of interest to researchers and scientists particularly in the theory of motion in mathematical physics. It should be noted here that results of special cases of system (1) where the deviating argument is absent or delay being zero have been investigated by few authors. (See, [12] and [8]).

Our motivation comes from [12]. Here, we consider a system of the form (1) using a suitable Lyapunov-Krasovskii's functional to obtain new satisfactory criteria for the convergence of solutions of system of delay differential equation. Analysis of such systems are quite difficult. The difficulty increases depending on the assumptions made on the nonlinear term with the deviating argument and the necessity for a complete Lyapunov-Krasovskii's functional. The Lyapunov-Krasovskii's functional for more satisfactory results are well discussed in [5]. The result obtained is not only new but will also contribute to the qualitative theory of systems nonlinear delay differential equations. A numeric example is given to show the application of the result and geometric display to describe the behavior of solutions under the criteria obtained.

We shall write equation (1) as

$$\dot{X} = H(Y)$$
$$\dot{Y} = -\Phi(X,Y)Y - \Psi(X) + \int_{t-\varrho(t)}^{t} \langle J_{\Psi}(X\sigma), H(Y\sigma)\rangle d\sigma + P(t,X,Y). \tag{2}$$

The following is our main result

2. STATEMENT OF RESULT

Theorem 1: Given that $\Psi(0) = H(0) = 0$, we assume that $\delta_H, \delta_P, \delta_S, \triangle_H, \triangle_P, \triangle_S > 0$ and $\delta_o > 0$ such that the following conditions hold:

(i) the Jacobian matrices $J_H(Y)$, $J_\Psi(X)$ and $\Phi(X,Y)$ exist, symmetric, commute pairwise and their eigenvalues satisfy

$$0 < \delta_H \leq \lambda_i(J_H(Y)) \leq \triangle_H$$
$$0 < \delta_S \leq \lambda_i(J_\Psi(X)) \leq \triangle_S$$
$$0 < \delta_P \leq \lambda_i(\Phi(X,Y)) \leq \triangle_P$$
$$(i = 1, 2..., n)$$

(ii) $P(t,X,Y)$ satisfies

$$\|P(t,X_2,Y_2) - P(t,X_1,Y_1)\| \leq \delta_o\{\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2\}^{\frac{1}{2}}$$

for any $X, Y \in \mathbb{R}^n$, provided

$$\xi < min\left\{\frac{(\alpha+\beta\delta_H)\delta_S}{(\alpha+\beta)\Delta_S}; \frac{(1-\upsilon)[(\delta_P-\alpha)\delta_H - \beta\delta_H\Delta_H]}{2\delta_H\Delta_S(1-\upsilon) + (\delta_H+\alpha+\beta)\Delta_S\delta_H\Delta_H}\right\}$$

and $\delta_o < \mu$,
then any two solutions $X_2(t)$, $X_1(t)$ and $Y_2(t)$, $Y_1(t)$ of (1) necessarily converge.

The function $V(X_t, Y_t)$

The Lyapunov-Krasovskii's functional $V(X_t, Y_t)$ that will be used to prove our result is defined by

$$\begin{aligned}2V(X_t, Y_t) &= 2\int_0^1 \langle \Psi(\vartheta X)X\rangle d\vartheta + 2\int_0^1 \langle H(\sigma Y)Y\rangle d\sigma \\ &+ 2\alpha\langle X, Y\rangle + 2\beta\langle X, H(Y)\rangle \\ &+ \Lambda\int_{-\varrho(t)}^{t}\int_{t+s}^{t}\langle H(Y\sigma), H(Y\sigma)\rangle d\sigma ds,\end{aligned} \tag{3}$$

for $0 < \alpha < 1$ and $0 < \beta < 0.1$.

Using (3) above, we establish the following result.

Lemma 1. Given that all the conditions on $H(Y), \Psi(X)$ and $\Phi(X, Y)$ in Theorem 1 hold. Then, we have $D_1 > 0$ and $D_2 > 0$ such that

$$D_1(\|X\|^2 + \|Y\|^2) \le 2V(X_t, Y_t) \le D_2(\|X\|^2 + \|Y\|^2) \tag{4}$$

for arbitrary $X, Y \in \mathbb{R}^n$.

Proof of Lemma 1:

Clearly $V(0, 0) = 0$ in (3) and $V(X_t, Y_t)$ in (3) after re-arrangement becomes

$$\begin{aligned}2V(X_t, Y_t) &= \langle \alpha^{\frac{1}{2}}X + \alpha^{\frac{1}{2}}Y, \alpha^{\frac{1}{2}}X + \alpha^{\frac{1}{2}}Y\rangle \\ &+ \langle \beta^{\frac{1}{2}}X + \beta^{\frac{1}{2}}H(Y), \beta^{\frac{1}{2}}X + \beta^{\frac{1}{2}}H(Y)\rangle \\ &+ 2\int_0^1 \langle G(\vartheta X)X\rangle d\vartheta - \alpha\langle X, X\rangle - \beta\langle X, X\rangle \\ &+ 2\int_0^1 \langle H(\sigma Y)Y\rangle d\sigma - \beta\langle H(Y), H(Y)\rangle - \alpha\langle Y, Y\rangle \\ &+ \Lambda\int_{-\varrho(t)}^{t}\int_{t+s}^{t}\langle H(Y\sigma), H(Y\sigma)\rangle d\sigma ds.\end{aligned}$$

For each of the term of the above function, we obtain the estimates by the same reasoning in [1], [2], [7] and (i) of Theorem 1, we have that the term

$$\begin{aligned}\langle H(Y), H(Y)\rangle &= \int_0^1 \frac{d}{d\sigma}\langle H(\sigma Y), H(Y)\rangle d\sigma \\ &= 2\int_0^1 \langle J_H(\sigma X), H(\sigma Y)\rangle d\sigma \\ &= 2\int_0^1\int_0^1 \varepsilon\langle J_H(\varepsilon Y)Y, H(\sigma\varepsilon Y)Y\rangle d\sigma d\varepsilon \\ &\geq \triangle_H \delta_H \|Y\|^2.\end{aligned}$$

Then,

$$\begin{aligned}2V(X_t, Y_t) \geq& (\delta_S - \alpha - \beta)\|X\|^2 + (\delta_H - \alpha - \beta \triangle_H \delta_H)\|Y\|^2 \\ &+ \|\beta^{\frac{1}{2}}X + \beta^{\frac{1}{2}}H(Y)\|^2 + \|\alpha^{\frac{1}{2}}X + \alpha^{\frac{1}{2}}Y\|^2 \\ &+ \Lambda\int_{-\varrho(t)}^{t}\int_{t+s}^{t}\langle H(Y\sigma), H(Y\sigma)\rangle d\sigma ds,\end{aligned} \tag{5}$$

where

$$\delta_S - \alpha - \beta > 0, \quad \delta_H - \alpha - \beta \triangle_H \delta_H > 0 \tag{6}$$

and the integral

$$\Lambda\int_{-\varrho(t)}^{t}\int_{t+s}^{t}\langle H(Y\sigma), H(Y\sigma)\rangle d\sigma ds > 0.$$

From (5), it is quite obvious that the function $V(X_t, Y_t)$ defined in (3) is definitely positive and $V(X_t, Y_t)$ also satisfy the conditions of [Theorem A and Theorem B, [4]] and followed the same reasoning in [20].

Hence, there is a $D_1 > 0$ small enough such that

$$2V(X_t, Y_t) \geq D_1(\|X\|^2 + \|Y\|^2).$$

The right side of inequality (4) of Lemma 1 follows by the same reasoning in [1], [2], [7] and (i) of Theorem 1 if we choose from (3), the term

$$2\int_0^1 \langle \Psi(\vartheta X)X\rangle d\vartheta = \int_0^1\int_0^1 \vartheta_1\langle J_\Psi(\vartheta_1\vartheta_2 X)X, X\rangle d\vartheta_1 d\vartheta_2$$

$$2\int_0^1 \langle H(\sigma Y)Y\rangle d\sigma = \int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_1 d\sigma_2$$

Thus,

$$
\begin{aligned}
\delta_S\|X\|^2 &\leq \int_0^t \langle \Psi(\vartheta X), X\rangle d\vartheta \leq \Delta_S\|X\|^2 \\
\delta_H\|Y\|^2 &\leq \int_0^t \langle H(\sigma Y), Y\rangle d\sigma \leq \Delta_H\|Y\|^2 \\
\langle X, H(Y)\rangle &= \int_0^t \langle J_H(\sigma Y)Y, X\rangle d\sigma \\
&\leq \int_0^t \langle J_H(\sigma Y)Y, X\rangle d\sigma \\
&\leq \Delta_H\|X\|\|Y\| \\
&\leq \frac{1}{2}\Delta_H\left(\|X\|^2 + \|Y\|^2\right) \\
|\langle X, Y\rangle| &\leq \|X\|\|Y\| \leq \left(\|X\|^2 + \|Y\|^2\right) \quad and
\end{aligned}
$$

$$
\Lambda \int_{-\varrho(t)}^t \int_{t+s}^t \langle H(Y\sigma), H(Y\sigma)\rangle d\sigma ds \leq \frac{1}{2}\Lambda\varrho^2(t)s\|Y\|^2.
$$

It follows that,

$$
\begin{aligned}
2V(X_t, Y_t) &\leq (2\Delta_G + \alpha + \beta\Delta_H)\|X\|^2 + (2\Delta_H + \alpha + \beta\Delta_H + \frac{1}{2}\Lambda\xi^2 s)\|Y\|^2 \\
&\leq D_2\left(\|X\|^2 + \|Y\|^2\right),
\end{aligned}
$$

where

$$
D_2 = \max\{2\Delta_S + \alpha + \beta\Delta_H; 2\Delta_H + \alpha + \beta\Delta_H + \frac{1}{2}\Lambda\xi^2 s\}.
$$

Hence, (4) of Lemma 1 is established where D_1, D_2 are finite constants.

Proof of Theorem 1

We take $(X_1(t), X_2(t)), (Y_1(t), Y_2(t)) \in \mathbb{R}^n$ to be any solution of (1) and let $\Omega = \Omega(t)$ be defined by

$$
\Omega(t) = V(X_2(t) - X_1(t), Y_2(t) - Y_1(t)),
$$

and V is the function earlier defined in (3) with X, Y substituted with $X_2 - X_1$ and $Y_2 - Y_1$ respectively.

Following Lemma 1, we have D_3 and D_4 such that

$$
D_3(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2) \leq \Omega(t) \leq D_4(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2). \tag{7}
$$

Next, differentiating $\Omega(t)$ with respect to t along the system (2) we obtain,

$$\dot{\Omega}(t) = -\Omega_1 - \Omega_2 + \Omega_3 + \Omega_4,$$

where

$$\begin{aligned}\Omega_1 &= \frac{1}{2}\langle(\alpha+\beta\delta_H)X_2 - X_1, \Psi(X_2) - \Psi(X_1)\rangle \\ &+ \frac{1}{2}\Big\{\langle(\delta_P-\alpha)Y_2 - Y_1, H(Y_2) - H(Y_1)\rangle - \beta\langle H(Y_2) - H(Y_1), H(Y_2) - H(Y_1)\rangle\Big\},\end{aligned}$$

$$\begin{aligned}\Omega_2 &= \frac{1}{2}\langle(\alpha+\beta\delta_H)X_2 - X_1, \Psi(X_2) - \Psi(X_1)\rangle \\ &+ \frac{1}{2}\Big\{\langle(\delta_P-\alpha)Y_2 - Y_1, H(Y_2) - H(Y_1)\rangle - \beta\langle H(Y_2) - H(Y_1), H(Y_2) - H(Y_1)\rangle\Big\} \\ &+ \langle(\alpha\delta_P + \beta\delta_P\delta_H)X_2 - X_1, Y_2 - Y_1\rangle,\end{aligned}$$

$$\begin{aligned}\Omega_3 &= \Lambda\varrho(t)\langle H(Y_2) - H(Y_1), H(Y_2) - H(Y_1)\rangle \\ &- \Lambda(1-\varrho'(t))\int_{t-\varrho(t)}^{t}\langle H(Y_2\sigma) - H(Y_1\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle \\ &+ \langle H(Y_2) - H(Y_1), \int_{t-\varrho(t)}^{t} J_\Psi(X_2 - X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma \\ &+ \langle\alpha(X_2 - X_1), \int_{t-\varrho(t)}^{t} J_\Psi(X_2 - X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma \\ &+ \langle\beta(X_2 - X_1), \int_{t-\varrho(t)}^{t} J_\Psi(X_2 - X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma\end{aligned}$$

and

$$\Omega_4 = [(\alpha+\beta\delta_H)\|X_2 - X_1\| + \delta_H\|Y_2 - Y_1\|]\|P(t, X_2, Y_2) - P(t, X_1, Y_1)\|.$$

The following estimate exist by the same argument in [3],

$$\begin{aligned}\Psi(X_2) - \Psi(X_1) &= \int_0^1 J_\Psi(X_2 - X_1)(r)dr \\ H(Y_2) - H(Y_1) &= \int_0^1 J_H(Y_2 - Y_1)(s)ds\end{aligned} \tag{8}$$

where $r = pX_1 + (1-p)X_2$, $0 \le p \le 1$ and $s = qY_1 + (1-q)Y_2$, $0 \le q \le 1$.

Using the hypotheses in Theorem 1 and (8), we get

$$\Omega_1 \geq \frac{1}{2}\langle(\alpha+\beta\delta_H)X_2 - X_1, \delta_\Psi(X_2 - X_1)\rangle$$
$$+\frac{1}{2}\Big\{\langle(\delta_P-\alpha)Y_2 - Y_1, \delta_H(Y_2-Y_1)\rangle - \beta\langle\delta_H\Delta_H(Y_2-Y_1),(Y_2-Y_1)\rangle\Big\}.$$

$$\Omega_1 \geq D_5(\|X_2-X_1\|^2 + \|Y_2-Y_1\|^2),$$

where $D_5 = \frac{1}{2}min\{(\alpha+\beta\delta_H)\delta_S, (\delta_P-\alpha)\delta_H - \beta\delta_H\Delta_H\}$.

$$\Omega_2 \geq \|l_1(\alpha\delta_P+\beta\delta_P\delta_H)(X_2-X_1) + 2^{-1}l_1^{-1}(Y_2-Y_1)\|^2$$
$$+\langle[2^{-1}(\alpha+\beta\delta_H)\delta_S - l_1^2(\alpha\delta_P+\beta\delta_P\delta_H)^2](X_2-X_1),(X_2-X_1)\rangle$$
$$+\langle\{[2^{-1}(\delta_P-\alpha)\delta_H - \beta\delta_H\Delta_H] - 4^{-1}l_1^{-2}\}(Y_2-Y_1),(Y_2-Y_1)\rangle$$

$$\Omega_2 \geq \|l_1(\alpha\delta_P+\beta\delta_P\delta_H)(X_2-X_1) + 2^{-1}l_1^{-1}(Y_2-Y_1)\|^2$$
$$+\{2^{-1}(\alpha+\beta\delta_H)\delta_S - l_1^2(\alpha\delta_P+\beta\delta_P\delta_H)^2\}\|X_2-X_1\|^2$$
$$+\{[2^{-1}(\delta_P-\alpha)\delta_H - \beta\delta_H\Delta_H] - 4^{-1}l_1^{-2}\}\|Y_2-Y_1\|^2.$$

If we set

$$l_1^2 = \frac{1}{2}min\left\{\frac{(\alpha+\beta\delta_H)\delta_S}{(\alpha\delta_P+\beta\delta_P\delta_H)^2}, \frac{1}{(\delta_P-\alpha)\delta_H - \beta\delta_H\Delta_H}\right\},$$

$$\Omega_2 \geq 0, \quad \forall\, X, Y \in \mathbb{R}^n.$$

In Ω_3, we give the estimates for the following and using the fact that $2|\langle P, Q\rangle| \leq (\|P\|^2 + \|Q\|^2)$

$$\langle H(Y_2) - H(Y_1), \int_{t-\varrho(t)}^{t} J_\Psi(X_2-X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma$$
$$\leq \frac{1}{2}\delta_H\Delta_S\varrho(t)\|Y_2-Y_1\|^2$$
$$+\frac{1}{2}\delta_H\Delta_S\int_{t-\varrho(t)}^{t}\langle H(Y_2\sigma) - H(Y_1\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma,$$

$$\langle \alpha(X_2-X_1), \int_{t-\varrho(t)}^{t} J_\Psi(X_2-X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma$$
$$\leq \frac{1}{2}\alpha\Delta_S\varrho(t)\|X_2-X_1\|^2$$
$$+\frac{1}{2}\alpha\Delta_S\int_{t-\varrho(t)}^{t}\langle H(Y_2\sigma) - H(Y_1\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma$$

and

$$\begin{aligned}&\langle \beta(X_2 - X_1), \int_{t-\varrho(t)}^{t} J_{\Psi}(X_2 - X_1)(\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma \\ &\leq \frac{1}{2}\beta\Delta_S\varrho(t)\|X_2 - X_1\|^2 \\ &+ \frac{1}{2}\beta\Delta_S \int_{t-\varrho(t)}^{t} \langle H(Y_2\sigma) - H(Y_1\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma.\end{aligned}$$

Thus

$$\begin{aligned}\Omega_3 &\leq \frac{1}{2}(\alpha + \beta)\Delta_S\varrho(t)\|X_2 - X_1\|^2 \\ &+ \frac{1}{2}(\delta_H\Delta_S + \Lambda\delta_H\Delta_H)\varrho(t)\|Y_2 - Y_1\|^2 \\ &+ (\delta_H\Delta_S + \alpha\Delta_S + \beta\Delta_S - 2\Lambda(1 - \varrho'(t)) \int_{t-\varrho(t)}^{t} \langle H(Y_2\sigma) - H(Y_1\sigma), H(Y_2\sigma) - H(Y_1\sigma)\rangle d\sigma.\end{aligned}$$

Using the assumption on $\varrho(t)$ and $\varrho'(t)$ and choosing

$$\Lambda = \frac{(\delta_H + \alpha + \beta)\Delta_S}{2(1 - \upsilon)} > 0,$$

we have that

$$\begin{aligned}\Omega_3 &\leq \frac{1}{2}(\alpha + \beta)\Delta_S\xi\|X_2 - X_1\|^2 \\ &+ \frac{1}{2}\left(2\delta_H\Delta_S + \frac{(\delta_H + \alpha + \beta)\Delta_S\delta_H\Delta_H}{2(1 - \upsilon)}\right)\xi\|Y_2 - Y_1\|^2.\end{aligned}$$

Now combining estimates $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 for $\dot{\Omega}(t)$, we get

$$\begin{aligned}\dot{\Omega}(t) \leq &- \frac{1}{2}\Big\{(\alpha + \beta\delta_H)\delta_S - (\alpha + \beta)\Delta_S\xi\Big\}\|X_2 - X_1\|^2 \\ &- \frac{1}{2}\Big\{(\delta_P - \alpha)\delta_H - \beta\delta_H\Delta_H - (2\delta_H\Delta_S + \frac{(\delta_H + \alpha + \beta)\Delta_S\delta_H\Delta_H}{2(1 - \upsilon)}\xi\Big\}\|Y_2 - Y_1\|^2 \\ &+ \Big\{(\alpha + \beta\delta_H)\|X_2 - X_1\| + \delta_H\|Y_2 - Y_1\|\Big\}\|P(t, X_2, Y_2) - P(t, X_1, Y_1)\|.\end{aligned}$$

If we choose

$$\xi < min\left\{\frac{(\alpha + \beta\delta_H)\delta_S}{(\alpha + \beta)\Delta_S}; \frac{(1 - \upsilon)[(\delta_P - \alpha)\delta_H - \beta\delta_H\Delta_H]}{2\delta_H\Delta_S(1 - \upsilon) + (\delta_H + \alpha + \beta)\Delta_S\delta_H\Delta_H}\right\},$$

we have

$$\dot{\Omega}(t) \leq -D_6(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2) + 2^{\frac{1}{2}}\delta_o D_7\Big\{(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2)^{\frac{1}{2}}(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2)^{\frac{1}{2}}\Big\},$$

where $D_6 > 0, D_7 > 0$ are finite constants.

$$\dot{\Omega}(t) \leq -(D_6 - 2^{\frac{1}{2}}\delta_o D_7)(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2)$$

So that

$$\dot{\Omega}(t) \leq -D_8(\|X_1 - X_2\|^2 + \|Y_1 - Y_2\|^2) \tag{9}$$

where $D_8 = (D_6 - 2^{\frac{1}{2}}\delta_o D_7)$ provided $\delta_o < \mu$ where μ is a sufficiently small positive constant and the fact that Ω basically satisfy the conditions of [Theorem A and Theorem B, [4]] and followed the same reasoning in [20].

In view of (7), (9) implies that

$$\dot{\Omega}(t) \leq -D_9\Omega, \tag{10}$$

where $D_9 = D_8 D_4^{-1} > 0$.

Integrating (10) between t_o and t, we have that

$$\Omega \leq \Omega(t_o)\exp(-D_9(t - t_o)), \;\; t \geq t_o$$

which implies that $\Omega(t) \to 0$ as $t \to \infty$.

By (7), it shows that

$$\|X_2 - X_1\| \to 0 \;\; and \;\; \|Y_2 - Y_1\| \to 0 \quad as \;\; t \to \infty. \quad \square$$

NUMERICAL EXAMPLE

Consider the system (1) for $n = 2$ in the form,

$$\begin{aligned} \dot{X} &= H(Y) \\ \dot{Y} &= -\Phi(X,Y)Y - \Psi(X(t - \varrho(t)) + P(t, X, Y), \end{aligned} \tag{11}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$

$H(Y) = \begin{pmatrix} 4y_1 + 2 \\ 6y_2 + 5 \end{pmatrix},$

$\Psi(X) = \begin{pmatrix} 3x_1(t - \varrho(t)) + \frac{0.002}{1+12x_1} \\ 4x_2(t - \varrho(t)) + \frac{0.002}{1+16x_2} \end{pmatrix},$

$\Phi(X,Y) = \begin{pmatrix} 2+x_1^2+y_1^2 & 0 \\ 0 & 3+x_2^2+y_2^2 \end{pmatrix}$

and $P(t,X,Y) = \begin{pmatrix} \frac{1}{1+t^2+x^2+y^2} \\ \frac{1}{2+t^2+x^2+y^2} \end{pmatrix}$.

Thus,

$J_H(X) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$

and $J_\Psi(X) = \begin{pmatrix} 12+3(-0.2+t) & 0 \\ 0 & 16+4(-0.2+t) \end{pmatrix}$.

Clearly, $J_H(X)$, $J_\Psi(X)$ and $\Phi(X,Y)$ are symmetric and commute pairwise. That is,

$$J_\Psi(X)\Phi(X,Y) = \Phi(X,Y)J_\Psi(X),$$
$$J_H(X)J_\Psi(X) = J_\Psi(X)J_H(X),$$
$$J_H(X)\Phi(X,Y) = \Phi(X,Y)J_H(X).$$

We obtain the eigenvalues of the matrices $J_H(X)$, $J_\Psi(X)$ and $\Phi(X,Y)$ as follows

$$\delta_H = 4 \le \lambda_i(J_H(Y)) \le 6 = \Delta_H \quad (i=1,2),$$
$$\delta_S = 12 \le \lambda_i(J_\Psi(X)) \le 16 = \Delta_S \quad (i=1,2)$$

and

$$\delta_P = 2 \le \lambda_i(\Phi(X,Y)) \le 3 = \Delta_P \quad (i=1,2).$$

We choose $\alpha = 0.4$, $\beta = 0.06$ and
$\delta_S - \alpha - \beta = 11.54 > 0, \quad \delta_H - \alpha - \beta \triangle_H \delta_H = 2.16 > 0.$

If we take $\varrho(t) = \frac{1}{2t^2+4}$, then $0 \le \frac{1}{2t^2+4} \le \xi$

and that $\varrho'(t) = \frac{-4t}{(2t^2+4)^2} \le \upsilon$. We choose $\upsilon = 0.5$, so that

$$\xi < min\left\{3, 0.24\right\}.$$

It follows that $|\varrho(t)| \le 0.20$, if the delay is increased beyond this range a discontinuous function of t appears which may lead to chaos. Thus, all the conditions of Theorem 1 are satisfied. Therefore, every solution $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\quad Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ of (1) are convergent as $t \to \infty$.

3. CONCLUSION

In this study, the convergence behaviour of solutions of certain system of second order nonlinear delay differential equation was carried out. New sufficient conditions on the convergence of solutions of the system of delay differential equation was established using the Lyapunov's direct method. Numerical and geometrical analysis were given in system (11) which satisfies all the conditions of Theorem 1 and (6). This new result significantly improves those available in the literature as well as contribute to the qualitative theory of systems nonlinear delay differential equations.

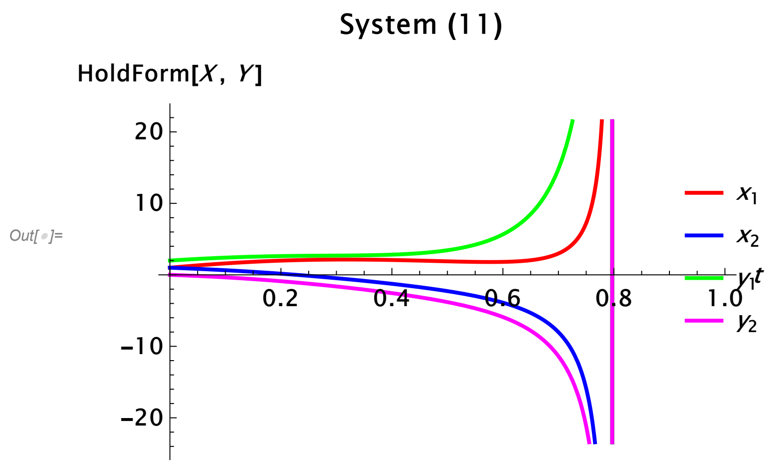

FIGURE 1. The plot of $X = \{x_1(t), x_2(t)\}$ (in red and blue) respectively and $Y = \{y_1(t), y_2(t)\}$ (in green and pink) respectively satisfying the conditions of Theorem 1 and (6) if $\varrho(t) = 0.1$ as $t \to \infty$.

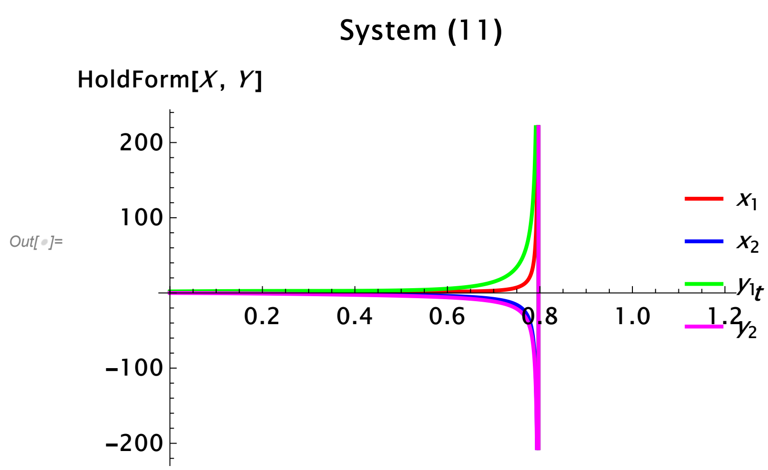

FIGURE 2. The plot of $X = \{x_1(t), x_2(t)\}$ (in red and blue) respectively and $Y = \{y_1(t), y_2(t)\}$ (in green and pink) respectively satisfying the conditions of Theorem 1 and (6) if $\varrho(t) = 0.24$ as $t \to \infty$.

Figure 1 and Figure 2 show that the solutions $X = \{x_1(t), x_2(t)\}$ and $Y = \{y_1(t), y_2(t)\}$ are convergent and in Figure 3 chaotic function of t appear for $\varrho(t) = 3.0$ and Figure 4 show the vector plot associated with the system (11) showing that the solutions are converging.

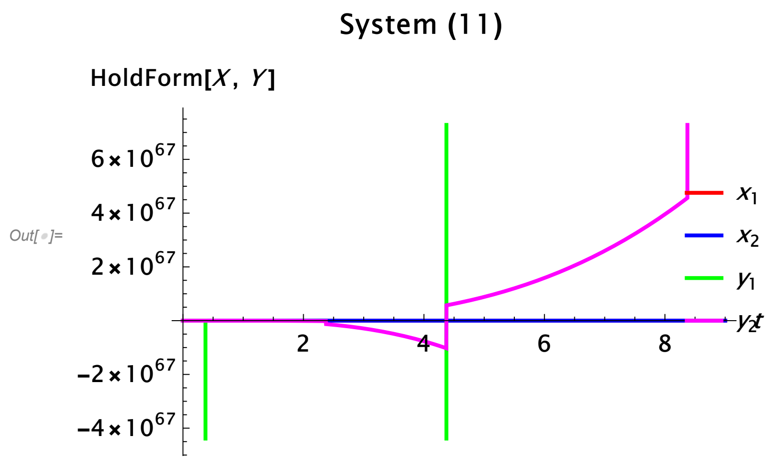

FIGURE 3. The solutions $X = \{x_1(t), x_2(t)\}$ and $Y = \{y_1(t), y_2(t)\}$ are discontinuous and appear to be chaotic if $\varrho(t) = 3.0$

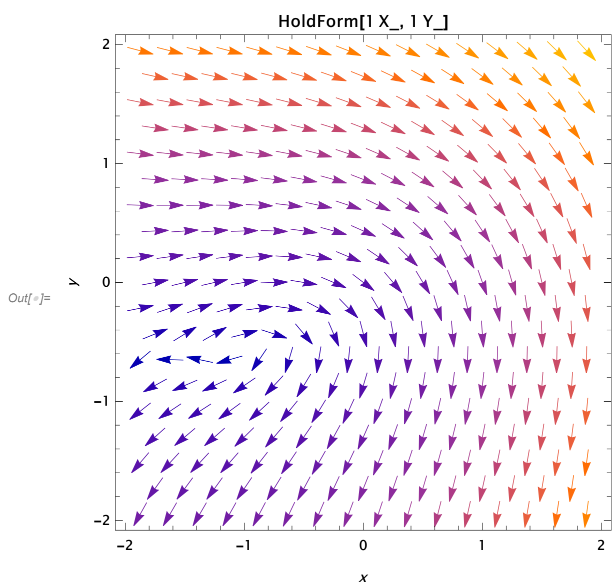

FIGURE 4. The vector plot associated with the system (11) satisfying the conditions of Theorem 1 and (6) together with several solutions of the system.

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