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VALUE OF A DIFFERENTIAL GAME PROBLEM WITH MULTIPLE PLAYERS IN A CERTAIN HILBERT SPACE

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ABSTRACT. We study differential game problem involving countable number of pursuers and one evader in the space l_2. Players' motion obey ordinary differential equations with integral constraints subjected to the control functions of the players. Termination time of the game is fixed. The payoff functional is the greatest lower bound of distances between pursuers and the evader when the game is terminated. Optimal strategies of the players are constructed and value of the game is found.

Keywords and phrases: Differential game; Value of the game; Optimal strategies.

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1. INTRODUCTION

There are numerous publications devoted to the study of pursuit-evasion differential game problem with multiple pursuers, for example [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14]. In some of these cited papers, motions of the players obey the differential equations:

$$\dot{x} = a(t)u(t), x(0) = x_0, \quad \dot{y} = b(t)v(t), y(0) = y_0, \tag{1}$$

where $u(t), v(t)$ are control functions of the players which are either subjected to geometric or integral constraints. The questions answered include but not limited to finding value of the game; conditions for completion of pursuit and construction of optimal strategies of the players.

The problems considered in [8], [10], [11] and [14] motions of the players obey the differential equations (1), the case where $a(t) =$

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$b(t) = 1$. Algorithm for determining the payoff function for all possible players' positions is constructed in [11]. Ivanov and Ledyaev [10] obtained sufficient conditions for finding optimal pursuit time in the space R^2, where control functions subject to geometric constraints. Optimal strategies of the players and value of the game is found in [8]. In [14], condition for capture of at least one evader is obtained, where the control functions for both pursuers and evaders are subjected to geometric constraints with state variable also constrained.

Optimal strategies of the players and sufficient conditions for optimality of pursuit time in the differential game problem with multiple pursuers described by (1) in the space R^2 are announced in [4]. In this study control function is subjected to geometric constraint.

In [6] differential game described by (1), where $a(t) = b(t) = 1$, of many pursuers and one evader is studied in the space l_2. Geometric constraints imposed on the control functions of the players. Optimal strategies of the players are constructed and value of the game is found.

Ibragimov and Mehdi in [5] studied pursuit-Evasion differential game with many inertial players described by (1), the case where $a(t) = b(t) = (\theta - t)$, in the space l_2. Control functions of the players are subjected to integral constraints(the case for geometric constraints is studied in [2]). Value of the game is found and optimal strategies of the players are constructed. This problem was also studied in [3] but with a different approach.

In the present paper, pursuit-evasion differential game problem with countable number of pursuers and one evader described by (1) is investigated in the space l_2. Termination time of the game is fixed. The payoff functional is the greatest lower bound of distances between pursuers and the evader when the game is terminated. Optimal strategies of the players are constructed and value of the game is found. The present work generalizes the problem considered in [5]. Therefore, the two papers are closely related in spirit.

2. STATEMENT OF THE PROBLEM

Consider $l_2 = \{\alpha = (\alpha_1, \alpha_2, \dots) : \sum_{k=1}^{\infty} \alpha_k^2 < \infty\}$, with inner product $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to R$ given by $\langle \alpha, \beta \rangle = \sum_{k=1}^{\infty} \alpha_k \beta_k$ and norm $|| \cdot || : l_2 \to [0, +\infty)$ given by $||\alpha|| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}$, $\alpha,\ \beta \in l_2$.

Let the motions of countably many pursuers $P_i,\ i = 1, 2, \dots$ and the evader E in the space l_2 be described by the equations

$$\begin{cases} P_i : \dot{x}_i = a(t)u_i,\ x_i(0) = x_{0i}, \\ E : \dot{y} = \ b(t)v, \quad y(0) = y_0, \end{cases} \tag{2}$$

where $x_i, x_{i0}, u_i, y, y_0, v \in l_2,\ u_i = (u_{i1}, u_{i2}, \dots)$ is a control parameter of the pursuer P_i and $v = (v_1, v_2, \dots)$ is that of the evader. Furthermore, $a(t)$ and $b(t)$ are scalar measurable functions such that $a(t) \ge b(t)$ for all $t \in [0, T]$ and let

$$A(T) = \int_0^T a^2(s)ds < \infty,\ B(T) = \int_0^T b^2(s)ds < \infty,$$

where T is fixed positive number which denotes duration of the game.

Let a ball (respectively, sphere) of radius r and center at point x_0 be denoted by $H(x_0, r) = \{x \in l_2 : ||x - x_0|| \le r\}$ (respectively, $S(x_0, r) = \{x \in l_2 : ||x - x_0|| = r\}$).

Definition 1. *A function $u_i(\cdot), u_i : [0, T] \to l_2$, such that $u_{ik} : [0, T] \to R^1, k = 1, 2, \dots,$ are Borel measurable functions and satisfies the inequality*

$$\left(\int_0^T \sum_{k=1}^{\infty} u_{ik}^2(s)ds \right)^{1/2} \le \rho_i,$$

(where ρ_i is given positive number) is an admissible control of the ith pursuer.

Definition 2. *A function $v(\cdot), u_i : [0,T] \to l_2$, such that $u_k : [0,T] \to R^1, k = 1,2,\dots,$ are Borel measurable functions and satisfies the inequality*

$$\left(\int_0^T \sum_{k=1}^{\infty} v_k^2(s)ds\right)^{1/2} \leq \sigma,$$

(where σ is given positive number) is an admissible control of the evader.

When the players' admissible control are chosen from given initial positions x_{0i} and y_0, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players are defined as

$$\begin{cases} x_i(t) = (x_{i1}(t), x_{i2}(t), \dots), \\ y(t) = (y_1(t), y_2(t), \dots), \end{cases} \tag{3}$$

where the coordinates are of the form

$$x_{ik}(t) = x_{0k} + \int_0^t a(s)u_{ik}(s)ds,\ y_k(t) = y_{0k} + \int_0^t b(s)v_k(s)ds.$$

It can be shown that $x_i(\cdot),\ y(\cdot) \in C(0,T;\ l_2)$, where $C(0,T;\ l_2)$ is the space of continuous functions

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots) \in l_2,\ t \geq 0,$$

such that

- $\alpha_k(t), 0 \leq t \leq T,\ k = 1,2,\dots$ are absolutely continuous functions
- $\alpha(t), 0 \leq t \leq T$, is a continuous function in the norm of l_2.

Definition 3. *A function $U_i(t, x_i, y, v),\ U_i : [0,\infty)\times l_2\times l_2\times l_2 \to l_2$, such that*

$$\begin{cases} \dot{x}_i = a(t)U_i(t, x_i, y, v),\ x_i(0) = x_{0i}, \\ \dot{y} = b(t)v,\ y(0) = y_0, \end{cases} \tag{4}$$

has a unique solution $(x_i(\cdot),\ y(\cdot))$, where $x_i(\cdot),\ y(\cdot) \in C(0,T; l_2)$, for any admissible control $v = v(t), 0 \leq t \leq T$ of the evader E is called the strategy of the pursuer P_i. A strategy U_i is deemed to be admissible if each of control involved in the formation of this strategy is admissible.

Definition 4. *Strategies U_{i0} of the pursuers $P_i, i = 1, 2, \dots$ are said to be optimal if*

$$\inf_{U_1, U_2, \dots} \Gamma_1(U_1, U_2, \dots) = \Gamma_1(U_{10}, U_{20}, \dots),$$

where $\Gamma_1(U_1, U_2, \dots) = \sup_{v(\cdot)} \inf_{i \in I} ||x_i(T) - y(T)||$; U_i are admissible strategies of the pursuers P_i, and $v(\cdot)$ is an admissible control of the evader E.

Definition 5. *A function $V(t, x_1, x_2, \dots, y)$, $V : [0, \infty) \times l_2 \times l_2 \cdots \to l_2$, such that*

$$\begin{cases} \dot{x}_i = a(t)u_i, \ x_i(0) = x_{0i}, i = 1, 2, \dots \\ \dot{y} = b(t)V(t, x_1, x_2, \dots, y), \ y(0) = y_0, \end{cases}$$

has a unique solution $(x_1(\cdot), x_2(\cdot), \dots, y(\cdot))$, with $x_i(\cdot)$, $y(\cdot) \in C(0, T; l_2)$, for arbitrary admissible control $u_i = u_i(t)$, $0 \le t \le T$, of the pursuers P_i, is called a strategy of the the evader E. The strategy V is said to be admissible when each of the control that formed the strategy is admissible.

Definition 6. *A strategy V_0 of the evader E is said to be optimal if* $\sup_V \Gamma_2(V) = \Gamma_2(V_0)$, *where* $\Gamma_2(V) = \inf_{u_1(\cdot), u_2(\cdot), \dots} \inf_{i \in I} ||x_i(T) - y(T)||$, *where $u_i(\cdot)$ are admissible strategies of the pursuers P_i.*

If $\Gamma_1(U_{10}, U_{20}, \dots) = \Gamma_2(V_0) = \gamma$, then we shall say the game has a value γ [13].

The problem is to find the optimal strategies U_{i0} and V_0 of the players P_i and E, respectively, and value of the game described by (2).

3. ATTAINABILITY DOMAINS

Consider the game described by (2). The attainability domain of the pursuers P_i at time T from initial position x_{i0} at a time $t_0 = 0$ is the closed ball $H(x_{i0}, A^{1/2}(T)\rho_i)$. To see that,

$$\begin{aligned}||x_i(T) - x_{i0}|| &= \left|\left|\int_0^T a(s)u_i(s)ds\right|\right| \\ &\leq \left(\int_0^T a^2(s)ds\right)^{1/2} \left(\int_0^T ||u_i(s)||^2 ds\right)^{1/2} \\ &\leq A^{1/2}(T)\rho_i.\end{aligned}$$

(here we used solution (3) of (2) and Cauchy-Schwartz inequality). On the other hand, if $\bar{x} \in H(x_{i0}, A^{1/2}(T)\rho_i)$, that is

$$||\bar{x} - x_{i0}|| \leq A^{1/2}(T)\rho_i,$$

then for the pursuer's control

$$u_i(t) = \frac{a(t)(\bar{x} - x_{i0})}{A(T)}, 0 \leq t \leq T,$$

guarantee the equality $x_i(T) = \bar{x}$. Indeed,

$$\begin{aligned}x_i(T) &= x_{i0} + \int_0^T a(s)u(s)ds \\ &= x_{i0} + \int_0^T a(s)\left(\frac{a(s)(\bar{x} - x_{i0})}{A(T)}\right) ds \\ &= x_{i0} + (\bar{x} - x_{i0}) = \bar{x}.\end{aligned}$$

The admissibility of the pursuers' control follows from the relations

$$\begin{aligned}\left(\int_0^T ||u_i(s)||^2 ds\right)^{1/2} &= \left(\int_0^T \frac{a^2(s)||\bar{x} - x_{i0}||^2}{A^2(T)} ds\right)^{1/2} \\ &\leq \frac{A^{1/2}(T)\rho_i}{A(T)} \left(\int_0^T a^2(s)ds\right)^{1/2} = \rho_i.\end{aligned}$$

Similarly, the attainability domain of the evader E at time T from the initial position y_0 at time $t_0 = 0$ is the closed ball $H(y_0, B^{1/2}(T)\sigma)$.

4. AUXILIARY GAME

In this section, we study the game with only one pursuer P_k, for $i = k$ and evader E . For simplicity, we drop the index k. This means we study the game described by

$$\begin{cases} P : \dot{x} = a(t)u, \ x(0) = x_0, \\ E : \dot{y} = b(t)v, \ y(0) = y_0, \end{cases} \tag{5}$$

with the state of the evader E satisfying the inclusion $y(T) \in X$, where the set X is defined as follows:

If $x_0 = y_0$, then

$$X = \left\{ \alpha \in l_2 : 2\langle \lambda, \ \alpha - y_0 \rangle \le A^{1/2}(T)\rho^2 \right\},$$

where λ is arbitrary fixed unit vector.

If $x_0 \neq y_0$ and for $\rho \ge \sigma$, then

$$X = \left\{ \alpha \in l_2 : 2\langle y_0 - x_0, \alpha \rangle \le B(T) \left(\rho^2 - \sigma^2 \right) + ||y_0||^2 - ||x_0||^2 \right\}.$$

In the game described by (5), the goal of the pursuer P is realize the equality $x(\tau) = y(\tau)$ at some $\tau \in [0, T]$ and that of the evader E is the opposite.

Lemma 1. *If $\rho \ge \sigma$ and $y(T) \in X$, then there exists pursuers strategy guaranteeing the equality $x(T) = y(T)$.*

Proof:
let $v = v(t)$ be an arbitrary admissible control of the evader E. We defined the pursuer's strategy as follows

$$U(t) = \begin{cases} \frac{b(t)}{a(t)} v(t), & if \ x_0 = y_0, \\ \frac{b(t)}{a(t)} \phi(t), & if \ x_0 \neq y_0, \ , \end{cases} \tag{6}$$

where

$$\phi(t) = \begin{cases} v(t) - \langle v(t), e \rangle e + e \sqrt{\frac{b^2(t)}{B(T)} (\rho^2 - \sigma^2) + \langle v(t), e \rangle^2}, & t \in [0, \tau] \\ v(t), & t \in (\tau, \ T], \end{cases}$$

$e = \frac{y_0 - x_0}{||y_0 - X_0||}$, $t \in [0, T]$ and τ is the instant time at which $x(\tau) = y(\tau)$ for the first time.

If $x_0 = y_0$, then using (6) it easy to deduce that $x(T) = y(T)$. We now let $x_0 \neq y_0$. Using the strategy (6), we have $y(t) - x(t) = e\eta(t)$, where

$$\begin{aligned}\eta(t) &= ||y_0 - x_0|| + \int_0^t b(s)\langle v(s), e\rangle ds \\ &\quad - \int_0^t b(s)\sqrt{\frac{b^2(s)}{B(T)}(\rho^2-\sigma^2) + \langle v(s), e\rangle^2 ds}. \qquad (7)\end{aligned}$$

We now claim that there exists a number $\tau \in [0, T]$ such that $\eta(\tau) = 0$. Since $\eta(0) = ||y_0 - x_0|| > 0$, to prove our claim it remains to show that $\eta(T) \leq 0$.

For the last integral in (7) we have

$$\int_0^T \sqrt{\frac{b^4(s)}{B(T)}(\rho^2-\sigma^2) + b^2(s)\langle v(t), e\rangle^2 ds} = \int_0^T |R(s)| ds$$

where $R(t) = \left(b^2(s)\sqrt{\frac{\rho^2-\sigma^2}{B(T)}},\ b(s)\langle v(t), e\rangle\right)$, i.e., a two dimensional vector function. But

$$\begin{aligned}\int_0^T |R(s)| ds &\geq \left|\int_0^T R(s) ds\right| \\ &= \left|\left(\int_0^T b^2(s)\sqrt{\frac{\rho^2-\sigma^2}{B(T)}} ds,\ \int_0^T b(s)\langle v(t), e\rangle ds\right)\right| \\ &= \left|\left(B^{1/2}(T)(\rho^2-\sigma^2)^{1/2},\ \int_0^T b(s)\langle v(t), e\rangle ds\right)\right| \\ &= \left(B(T)(\rho^2-\sigma^2) + \left(\int_0^T b(s)\langle v(t), e\rangle ds\right)^2\right)^{1/2}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\eta(T) &\leq ||y_0 - x_0|| + \int_0^T b(s)\langle v(t), e\rangle ds \\ &\quad - \left(B(T)\left(\rho^2-\sigma^2\right) + \left(\int_0^T b(s)\langle v(s), e\rangle ds\right)^2\right)^{1/2}. \qquad (8)\end{aligned}$$

But by our assumption that $y(T) \in X$, for $x_0 \neq y_0$, we have

$$2\langle y_0 - x_0, y(T)\rangle \leq B(T)\left(\rho^2-\sigma^2\right) + ||y_0||^2 - ||x_0||^2,$$

from which we can deduce that

$$\langle e, y(T)\rangle \leq \frac{B(T)\left(\rho^2-\sigma^2\right)+||y_0||^2-||x_0||^2}{2||y_0-x_0||}. \tag{9}$$

Now using the definition (3) of $y(T)$ we have

$$\begin{aligned}\langle e, y(T)\rangle &= \left\langle e,\ y_0+\int_0^T b(s)v(s)ds\right\rangle \\ &= \langle e,\ y_0\rangle + \int_0^T b(s)\langle v(s),\ e\rangle ds,\end{aligned}$$

and using this and (9), we have

$$\int_0^T b(s)\langle v(s),\ e\rangle \leq \Lambda - \langle y_0,\ e\rangle, \tag{10}$$

where

$$\Lambda = \frac{B(T)\left(\rho^2-\sigma^2\right)+||y_0||^2-||x_0||^2}{2||y_0-x_0||}.$$

Since $B(T)\left(\rho^2-\sigma^2\right) \geq 0$ (i.e., $\rho > \sigma$ and $B(T) > 0$), then it follows from (8) and (10) that

$$\begin{aligned}\eta(T) \leq\ & ||y_0-x_0|| + \Lambda - \langle y_0,\ e\rangle \\ & - \left(B(T)\left(\rho^2-\sigma^2\right) + \left(\Lambda - \langle y_0,\ e\rangle\right)^2\right)^{1/2}.\end{aligned} \tag{11}$$

We now show that the right-hand side of the last inequality is equal to zero. That is

$$\begin{aligned}& ||y_0-x_0|| + \Lambda - \langle y_0,\ e\rangle \\ & = \left(B(T)\left(\rho^2-\sigma^2\right) + \left(\Lambda - \langle y_0,\ e\rangle\right)^2\right)^{1/2}.\end{aligned} \tag{12}$$

The expression in the left hand side of this equation is positive, we can therefore square both sides of this equation and obtain

$$\begin{aligned}& ||y_0-x_0||^2 + \left(\Lambda - \langle y_0,\ e\rangle\right)^2 + 2||y_0-x_0||\left(\Lambda - \langle y_0,\ e\rangle\right) \\ & = B(T)\left(\rho^2-\sigma^2\right) + \left(\Lambda - \langle y_0,\ e\rangle\right)^2.\end{aligned}$$

This implies that

$$\begin{aligned}& ||y_0-x_0||^2 + 2||y_0-x_0||\left(\frac{B(T)\left(\rho^2-\sigma^2\right)+||y_0||^2-||x_0||^2}{2||y_0-x_0||} - \langle y_0,\ e\rangle\right) \\ & = B(T)\left(\rho^2-\sigma^2\right).\end{aligned}$$

This equality is true since

$$||y_0 - x_0||^2 + ||y_0||^2 - ||x_0||^2 - 2||y_0 - x_0||\langle y_0,\ e\rangle = 0$$

This means that $\eta(T) \leq 0$. Therefore, a number $\tau \in [0,\ T]$ exists such that $\eta(\tau) = 0$. Hence, $y(\tau) - x(\tau) = 0$. Furthermore, for $t \in (\tau,\ T]$ and using the strategy (6) i.e., $u(t) = \frac{b(t)}{a(t)}v(t)$, we have

$$x(T) = x(\tau) + \int_\tau^T a(s)u(s)ds = y(\tau) + \int_\tau^T b(s)v(s)ds = y(T).$$

The proof of the lemma is complete.

5. MAIN RESULT

In this section, we construct optimal strategies of the players and value of the game described by (2).

Let

$$\gamma = \inf\left\{ l \geq 0 : H\left(y_0,\ \sqrt{B(T)}\sigma\right) \subset \bigcup_{i \in I}^{\infty} H\left(x_{i0},\ \sqrt{A(T)}\rho_i + l\right)\right\}, \tag{13}$$

where $I = \{1, 2, \ldots\}$.

Suppose that the following assertion is true

Assertion 1. *There exists a nonzero vector p_0 such that $\langle y_0 - x_{i0},\ p_0\rangle \geq 0$ for all $i \in I$.*

Theorem 1. *. If the **Assertion 1** holds and $\sigma \leq \rho_i + \gamma A^{-1/2}(T)$ for all $i \in I$, then the number γ defined by (13) is the value of the game (2).*

We now state some lemmas that are going to be used in the proof of the theorem above. Consider the sphere $S(y_0,\ r)$ and finitely or countably many balls $H(x_{i0},\ R_i)$ and $H(y_0,\ r)$, where $x_{i0} \neq y_0$ and r and $R_i, i \in I$ are positive numbers.

Lemma 2. *(see [6]). Let the se X_i be defined as follows:*
i. If $x_{i0} = y_0$ then

$$X_i = \{z \in l_2 : 2\langle z - y_0,\ \lambda_0\rangle \leq R_i\}$$

ii. If $x_{i0} \neq y_0$, then

$$X_i = \left\{ z \in l_2 : 2\langle y_0 - x_{i0},\ z\rangle \leq R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2 \right\}.$$

If Assertion 1 is true and $H(y_0,\ r) \subset \bigcup_{i\in I} H(x_{i0},\ R_i)$ then $H(y_0,\ r) \subset \bigcup_{i\in I} X_i$.

Lemma 3. *(see [6]). Let $\inf_{i\in I} R_i = R_0 > 0$. If Assertion 1 is true and for any $0 < \epsilon < R_0$ the set $\bigcup_{i\in I} H(x_{i0},\ R_i - \epsilon)$ does not contain the ball $H(y_0, r)$, then there exists a point $\bar{y} \in S(y_0,\ r)$ such that $||\bar{y} - x_{i0}|| \geq R_i$ for all $i \in I$.*

Proof of Theorem 1:

To prove this theorem we need to construct strategies of the those players that will satisfy the following inequalities:

$$\sup_{v(\cdot)} \inf_{i\in I} ||y(T) - x_i(T)|| \leq \gamma \leq \inf_{u_1,u_2,(\cdot)\dots} \inf_{i\in I} ||y(T) - x_i(T)||, \quad (14)$$

where $v(\cdot)$ is an arbitrary admissible control of the evader, and $u_1(\cdot), u_2(\cdot)\dots$ are an arbitrary admissible controls of the pursuers.

To prove the inequalities in (14), we first show the left hand inequality, which means that the value γ defined by (13) is guaranteed for the pursuers. To do that, we start by constructing strategies of the pursuers.

Let us introduce dummy pursuers $z_i,\ i \in I$, whose motions are described by the equations

$$\dot{z}_i = a(t)w_i^\epsilon,\ z_i(0) = x_{i0},$$

and whose controls must satisfy the inequality

$$\left(\int_0^T ||(w_{ik}^\epsilon(s))||^2 ds \right)^{1/2} \leq \bar{\rho}_i(\epsilon),$$

where $\bar{\rho}_i(\epsilon) = \rho_i + \gamma A^{-1/2}(T) + \frac{\epsilon}{k_i} A^{-1/2}(T),\ k = \max\{1, \rho_i\}$, $\epsilon \in (0, 1)$.

It can be shown that the attainability domain of the dummy pursuers z_i at time T from initial position x_{i0} is the ball

$$H\left(x_{i0},\ \bar{\rho}_i(\epsilon) A^{1/2}(T)\right).$$

We now construct the strategies of the dummy pursuers z_i as follows:

$$w_i^{\epsilon}(t) = \begin{cases} \frac{b(t)}{a(t)} v(t), & if\ x_{i0} = y_0, \\ \frac{b(t)}{a(t)} \phi_i(t), & if\ x_{i0} \neq y_0,, \end{cases} \tag{15}$$

where

$$\phi_i(t) = \begin{cases} v(t) - \langle v(t), e_i \rangle e_i + e_i \sqrt{\frac{b^2(s)}{B(T)} (\bar{\rho}_i^2(\epsilon) - \sigma^2) + \langle v(t), e_i \rangle^2}, & t \in [0,\ \tau_i] \\ v(t), & t \in (\tau_i, T]; \end{cases}$$

$e_i = \frac{y_0 - x_{i0}}{||y_0 - x_0||}$; $t \in [0, T]$ and τ_i is the instant time at which $z_i(\tau_i) = y(\tau_i)$ for the first time if it exists.

We now show that the strategy (15) is admissible.
i. For $x_{i0} = y_0$, and $0 \leq t \leq T$:

$$\begin{aligned} \left(\int_0^T ||w_i^{\epsilon}(s)||^2 ds \right)^{1/2} &= \left(\int_0^T \frac{b^2(s)}{a^2(s)} ||v(s)||^2 ds \right)^{1/2} \\ &\leq \left(\int_0^T ||v(s)||^2 ds \right)^{1/2} \\ &\leq \sigma \leq \rho_i + \gamma A^{-1/2}(T) \\ &\leq \rho_i + \gamma A^{-1/2}(T) + \frac{\epsilon}{k_i} A^{-1/2}(T) \\ &= \bar{\rho}_i(\epsilon). \end{aligned}$$

ii. For $x_{i0} \neq y_0$, and $0 \leq t \leq T$:

$$\begin{aligned} \int_0^T ||w_i^{\epsilon}(s)||^2 ds &= \int_0^{\tau_i} ||w_i^{\epsilon}(s)||^2 ds + \int_{\tau_i}^T ||w_i^{\epsilon}(s)||^2 ds \\ &\leq \int_0^{\tau_i} ||v(s)||^2 ds + \frac{\bar{\rho}_i^2(\epsilon) - \sigma^2}{B(T)} \int_0^{\tau_i} b^2(s) ds \\ &+ \int_{\tau_i}^T ||v(s)||^2 ds \\ &\leq \int_0^T ||v(s)||^2 ds + \frac{\bar{\rho}_i^2(\epsilon) - \sigma^2}{B(T)} \int_0^T b^2(s) ds \\ &\leq \sigma^2 + \bar{\rho}_i^2(\epsilon) - \sigma^2 = \bar{\rho}_i^2(\epsilon). \end{aligned}$$

We now use these strategies of the dummy pursuers to construct the strategies of the pursuers x_i

$$U_i(t) = \frac{\rho_i}{\bar{\rho}_i(0)} w_i^0(t), \ 0 \leq t \leq T. \tag{16}$$

It is cheap to show that this strategy is admissible.

We now show that these strategies ensure the left hand inequality in (14) as follows:
Considering Assertion 1 and the fact that

$$H\left(y_0, \ B^{1/2}(T)\sigma\right) \subset \bigcup_{i \in I}^{\infty} H\left(x_{i0}, \ A^{1/2}(T)\rho + \gamma + \frac{\epsilon}{k_i}\right),$$

(see definition of γ and lemma 2) we have

$$H\left(y_0, \ B^{1/2}(T)\sigma\right) \subset \bigcup_{i \in I}^{\infty} X_i^{\epsilon},$$

where the set X_i^{ϵ} is define by
i. for $x_{i0} = y_0$,

$$X_i^{\epsilon} = \left\{ z \in l_2 : 2\langle z - y_0, \ \lambda_0 \rangle \leq \sqrt{A(T)}\rho + \gamma + \frac{\epsilon}{k_i} \right\},$$

ii. for $x_{i0} \neq y_0$

$$X_i^{\epsilon} = \left\{ z \in l_2 : 2\langle y_0 - x_{i0}, \ z \rangle \leq \Omega \right\},$$

where $\Omega = \left(\sqrt{A(T)}\rho + \gamma + \frac{\epsilon}{k_i}\right)^2 - B(T)\sigma^2 + ||y_0||^2 - ||x_{i0}||^2.$

Consequently, the point $y(T) \in H\left(y_0, \ B(T)^{1/2}\sigma\right)$ belongs to some half space X_s^{ϵ} for some $i = s \in I$ and s depends on ϵ.

Since $\bar{\rho}_i(\epsilon) > \sigma$ (assumption of the theorem and definition of $\bar{\rho}_i(\epsilon)$); then it follows from the Lemma (1) that if the dummy pursuer z_s uses the strategy (15), then $z_s(T) = y(T)$. With this and considering the strategy (16) we have

$$\begin{aligned}
||y(T) - x_s(T)|| &= ||z_s(T) - x_s(T)|| \\
&= \left|\left|\int_0^T a(t)\left(w_s^\epsilon(t) - \frac{\rho_s}{\bar{\rho}_s(0)} w_s^0(t)\right) dt\right|\right| \\
&\leq \int_0^T ||a(t)(w_s^\epsilon(t) - w_s^0(t))|| dt \\
&+ \int_0^T \left|\left| a(t)\left(w_s^0(t) - \frac{\rho_s}{\bar{\rho}_s(0)} w_s^0(t)\right)\right|\right| dt.
\end{aligned}$$

We now estimate the integrals in the last inequality. Let us show that

$$\lim_{\epsilon \to 0} \sup_{i \in I} \int_0^T ||a(t)\left(w_i^\epsilon(t) - w_i^0(t)\right)|| dt = 0, \tag{17}$$

If $x_{i0} = y_0$, then by (15) $w_i^\epsilon(t) = v(t) = w_i^0(t)$ and obviously (17) holds. Similarly if $x_{i0} \neq y_0$, using (15) and with assumption that $\tau_i \in [0,\ T]$ exists then

$$\int_0^{\tau_i} ||w_i^\epsilon(t) - w_i^0(t)||^2 dt$$

$$\begin{aligned}
&= \int_0^{\tau_i} \frac{b^2(t)}{a^2(t)} \left(\sqrt{\varrho(\epsilon) + \langle v(t),\ e_i\rangle^2} - \sqrt{\varrho(0) + \langle v(t),\ e_i\rangle^2}\right)^2 dt \\
&\leq \int_0^{\tau_i} \left(\sqrt{\varrho(\epsilon) + \langle v(t),\ e_i\rangle^2} - \sqrt{\varrho(0) + \langle v(t),\ e_i\rangle^2}\right)^2 dt \\
&\leq \int_0^{\tau_i} \left(\sqrt{\varrho(\epsilon)} - \sqrt{\varrho(0)}\right)^2 dt \\
&\leq \int_0^T \frac{b^2(t)}{B(T)} \left(\sqrt{\bar{\rho}_i^2(\epsilon) - \sigma^2} - \sqrt{\bar{\rho}_i^2(0) - \sigma^2}\right)^2 dt \\
&= \left(\sqrt{\bar{\rho}_i^2(\epsilon) - \sigma^2} - \sqrt{\bar{\rho}_i^2(0) - \sigma^2}\right)^2 \\
&= \left(\sqrt{2\bar{\rho}_i(0)\frac{\epsilon}{k_i\sqrt{A(T)}} + \frac{\epsilon^2}{k_i^2 A(T)} + \bar{\rho}_i^2(0) - \sigma^2} - \sqrt{\bar{\rho}_i^2(0) - \sigma^2}\right)^2
\end{aligned}$$

$$\leq 2\bar{\rho}_i(0)\frac{\epsilon}{k_i\sqrt{A(T)}} + \frac{\epsilon^2}{k_i^2 A(T)}$$
$$\leq \left(2A^{-1/2}(T) + (2\gamma+1)\, A^{-1}(T)\right)\epsilon.$$

Here we use the notation $\varrho(\epsilon) = \dfrac{b^2(t)}{B(T)}\left(\bar{\rho}_i^2(\epsilon) - \sigma^2\right)$; the fact that $\epsilon \in (0,\ 1); k = \max[1,\ \rho_i]$ and the inequality

$$\frac{\bar{\rho}_i^2(0)}{k_i} \leq 1 + \gamma A^{-1/2}(T).$$

The estimation of the first integral is as follows

$$\int_0^T ||a(t)\left(w_i^\epsilon(t) - w_i^0(t)\right)||dt$$
$$= \int_0^{\tau_i} ||a(t)\left(w_i^\epsilon(t) - w_i^0(t)\right)||dt$$
$$\leq \left(\int_0^{\tau_i} a^2(t)dt\right)^{1/2}\left(\int_0^{\tau_i} ||\left(w_i^\epsilon(t) - w_i^0(t)\right)||^2 dt\right)^{1/2}$$
$$\leq \mu\epsilon,$$

for some positive number μ. We have the equality (17) holding for the case $x_0 \neq y_0$.

The second integral can be estimated as follows:

$$\int_0^T \left|\left|a(t)\left(w_s^0(t) - \frac{\rho_s}{\bar{\rho}_s(0)}w_s^0(t)\right)\right|\right| dt$$
$$= \left(1 - \frac{\rho_s}{\bar{\rho}_s(0)}\right)\int_0^T \left|\left|a(t)w_s^0(t)\right|\right| dt$$
$$\leq \left(1 - \frac{\rho_s}{\bar{\rho}_s(0)}\right)\left(\int_0^T a^2(t)dt\right)^{1/2}\left(\int_0^T \left|\left|w_s^0(t)\right|\right|^2 dt\right)^{1/2}$$
$$\leq \left(1 - \frac{\rho_s}{\bar{\rho}_s(0)}\right) A^{1/2}(T)\bar{\rho}_s(0) = \gamma.$$

Therefore, we have

$$||y(T) - x_s(T)|| \leq \mu\epsilon + \gamma.$$

From this, the inequality (14) follows.

We now prove the inequality in the right hand of (14) which will implies that the value γ is guaranteed for the the evader.

Now if $\gamma = 0$ there is nothing to construct because any admissible control of the evader can do the job. Therefore we let $\gamma > 0$.

For any $\epsilon > 0$, we have

$$H\left(y_0,\ B^{1/2}(T)\right) \not\subset \bigcup_{i=1}^{\infty} H\left(x_{i0},\ A^{1/2}(T)\rho_i + \gamma - \epsilon\right),$$

(see (13)). Then, by Lemma 3 there exists a point $\bar{y} \in S(y_0,\ B^{1/2}(T))\sigma$ or $||\bar{y} - y_0|| = B^{1/2}(T)\sigma$ such that

$$||\bar{y} - x_{i0}|| \geq A^{1/2}(T)\rho_i + \gamma.$$

With this and the fact that

$$||x_{i0}(T) - x_{i0}|| \leq \left(\int_0^T a^2(t)dt\right)^{1/2} \left(\int_0^T ||u_i(t)||^2\, dt\right)^{1/2} \leq A^{1/2}(T)\rho_i,$$

we deduce that

$$\begin{aligned} ||\bar{y} - x_i(T)|| &\geq ||\bar{y} - x_{i0}|| - ||x_i(T) - x_{i0}|| \\ &\geq A^{1/2}(T)\rho_i + \gamma - A^{1/2}(T)\rho_i \\ &= \gamma. \end{aligned}$$

Proving $\bar{y} = y(T)$ then the right hand inequality of (14) will follow. Now if the evader uses the control

$$v(t) = B^{-1/2}(T)\sigma b(t)\bar{e}, \quad 0 \leq t \leq T,$$

where $\bar{e} = \frac{\bar{y} - y_0}{||\bar{y} - y_0||}$, then

$$\begin{aligned} y(T) &= y_0 + \int_0^T b(t)v(t)dt \\ &= y_0 + \int_0^T b^2(t)B^{-1/2}(T)\sigma e dt \\ &= y_0 + B^{1/2}(T)\sigma \frac{\bar{y} - y_0}{||\bar{y} - y_0||} = \bar{y}. \end{aligned}$$

This completes the prove of the theorem.

6. DISCUSSIONS

The Problems studied in [5] and [3] are exactly the same and in each of the paper, value of the game was found. However, the condition $\rho > \sigma$, which is required in finding the value of the game in the former paper, is dropped in the later paper. As a result of this, the optimal strategies of the players constructed in the two papers are different.

Equations of motion of the players in game studied in [3] and [5] are special cases of that considered in this paper. In this work, we consider arbitrary functions $a(t)$ and $b(t)$ in place of the linear function $(\theta - t)$ considered in the last two cited papers. These functions are arbitrary but are required to satisfy the inequality $a(t) \geq b(t)$ for all $t \in [0, T]$.

The result of this paper can be implemented not only to the problem in [5] but also to many differential game problems. For example, it can be implemented for a differential game problem for which equations of motion of the players is described by

$$\begin{cases} P_i : \dot{x}_i = (\cos t)u_i, \ x_i(0) = x_{0i}, \\ E : \dot{y} = \ (\sin t)v, \ \ y(0) = y_0, \end{cases}$$

where $t \in [0, \frac{\pi}{4}]$. In particular, a game of fixed duration with $T = \frac{\pi}{4}, \ a(t) = \cos t$ and $b(t) = \sin t$.

Furthermore, the optimal strategies of the pursuers constructed in this paper are modified popular p-strategy used in [5].

7. CONCLUSION

We obtained value of a differential game problem with multiple pursuer and one evader in the Hilbert space l_2 under certain conditions. Optimal strategies of the players were constructed.

The problem considered in this paper is a further research recommended in the paper [5]. We employed method of proof that requires use of the solution of an auxiliary differential game in proving the main result. This approach was also used in many papers (see, for example [3], [5], [10] and [12]).

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REFERENCES

[1] W. Chodun, *Differential Games of Evasion with Many Pursuers*, Journal of Mathematical Analysis and Applications, **142** 370 - 389, 1989.

[2] G. Ibragimov and N. A. Hussin, *A Pursuit-Evasion Differential Game with Many Pursuers and One Evader*, Malaysian Journal of Mathematical Sciences **4**(2) 183-194, 2010.

[3] G. Ibragimov, A. N. Rasid, A. Kuchkurov and F. Ismail, *Multi Pursuer Differential Game of Optimal Approach with Integral Constraints on Controls Of Players*, Taiwanese Journal of Mathematics, **19**(3) 963-976, 2015.

[4] G. I. Ibragimov and B. B. Rikhsiev, *On some Sufficient Conditions for Optimalty of the Pursuit Time in the Differential Game with Multiple Pusuers*, Automation and Remote Control **67**(4) 529-537, 2006.

[5] G. I. Ibragimov and M. Salimi, *Pursuit-Evasion Differential Game with Many Inertial Players*, Mathematical Problems in Engineering **doi:10.1155/2009/653723** 2009.

[6] G. I. Ibragimov, *Optimal pursuit with countably many pursuers and one evader*, Differential Equations **41**(5) 627-635, 2005.

[7] G. Ibragimov and N. Satimov, *A Multiplayer Pursuit Differential Game on a Convext Set with Integral Constraints*, Astract and Applied Analysis **doi: 10.1155/2012/460171**, 2012.

[8] G. I. Ibragimov, *A Game of Optimal Pursuit of One Object by Several*, Journal of Applied Mathematics and Mechanics **62**(2) 187 -192, 1988.

[9] G. I. Ibragimov, *On a Multiperson Pursuit Problem with Integral Constraints on the Controls of the Players*, Mathematical Notes **70**(2) 201-212, 2001 .

[10] R. P. Ivanov and Yu. S. Ledyaev, *Optimality of pursuit time in a simple motion differential game of many objects*, Trudy Matematicheskogo Instituta imeni V. A. Steklova, **158** 87-97, 1981.

[11] A. Yu. Levchenkov and A. G. Pashkov, *A Game of Optimal Pursuit of One Non-inertial Object by Two Inertial Objects*, Prikl. Mekhan **49**(4) 536-547, 1985.

[12] A. Yu. Levchenkov and A. G. Pashkov, *Differential Game of Optimal approach of two inertial pursuers to a noninertial evader*, Journal of Optimazation Theory and application **65**(3), 501-517, 1990.

[13] A. I. Subbotin and A. G. Chentsov, *Optimazation of Guaranteed Result in Control Problems*, Moscow, Russia, 1981.

[14] D. A. Vagin and N. N. Petrov, *A Problem of Group Pursuit with Phase Constraints*, Journal of Applied Mathematics and Mechanics, **66** (2) 225-232, 2002.

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