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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper investigates the asymptotic behaviour of solutions of some second-order nonlinear ordinary, delay, and stochastic differential equations. The order of these differential equations are reduced to a system of first order and employed to construct a suitable complete Lyapunov functions and functional. Standard conditions are imposed on the nonlinear terms to obtain criteria that guarantee the asymptotic behaviour of solutions of the considered equations. Examples are given to illustrate the obtained results. Our results improve and extend some well-known results in the literature.

1. INTRODUCTION

The study of problems that involve the behaviour of solutions of ordinary differential equations (ODEs), delay or functional differential equations (DDEs or FDEs), and stochastic differential equations (SDEs) has been dealt with by many outstanding authors; see, for instance, Ademola and Ogundiran [7], Arnold [11], Balakrishnam [13], Burton [14–16], Hale [21–23], Itô and Nishio [24], Kellert *et al.* [25], Lasalle [26], Liu and Raffoul [28], Mao [31] and Yoshizawa [37].

Many differential equations of second order have been derived since the beginning of the 18th century as models for problems of classical mechanics and other fields of science. Consequently, the study of problems involving the asymptotic behaviour, existence and uniqueness of solutions of ordinary differential equations, delay differential equations and stochastic differential equations have been dealt with by several authors such as Abou-El-Ela *et al.*[1, 2], Ademola [3], Ademola *et al.*[4–6], Alaba and Ogundare [8, 9], Cartwright and Littlewood [17], Ezeilo [18, 19], Levinson [27], Omeike *et al.* [32], Tejumola [33], Tunç [34], Yeniçerioğlu [35, 36] and so on to mention but a few.

From the literature, the most often used method to study the asymptotic behaviour of solutions of differential equations is the second method of Lyapunov. See for instance, Max [31], Lyapunov [29, 30], Itô and Nishio [24]. The major advantage of this method is that the qualitative behaviour of solutions can be obtained and discussed without any prior knowledge of the solutions. However, the construction of the Lyapunov

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functions remains a general problem due to the lack of a unique way of constructing the functions.

So far, the study of second-order nonlinear ordinary differential equations, delay differential equations and stochastic differential equations has led to some interesting results. For instance, Graef *et al.* [20] studied the asymptotic behaviour of solutions of a second-order nonlinear differential equation

$$(a_1(t)x')' + q_1(t)f_1(x)g_1(x') = r_1(t).$$

In another interesting paper, Tunç [34] established sufficient criteria for the stability and boundedness of solutions of non-autonomous differential equation of second order

$$x'' + a_2(t)[f_2(x,x')x' + g_2(x,x')x']x' + b_2(t)h_2(x) = e_2(t,x,x').$$

Alaba and Ogundare [8] worked on the asymptotic behaviour of solutions of certain second-order ordinary differential equations of the form

$$x'' + a_3(t)f_3(x,x')x' + b_3(t)g_3(x) = e_3(t,x,x').$$

Next, we shift our attention to second-order nonlinear delay differential equations, Ademola *et al.* [4] worked on the periodicity, stability, and boundedness of solutions to the following nonlinear delay differential equation

$$x''(t) + \phi_4(t)f_4(x(t),x(t-\rho(t)),x'(t),x'(t-\rho(t))) + g_4(x(t-\rho(t))) = p_4(\cdot),$$

where $p_4(\cdot) = p_4(t,x(t),x'(t))$. We now consider second-order nonlinear stochastic differential equations with Abou-El-Ela *et al.* [1, 2] where new results on stability and boundedness of solutions of the equation of the type

$$x''(t) + g_5(x'(t)) + bx(t-h) + \vartheta x(t)\theta'(t) = p_5(t,x(t),x'(t-h))$$

was acquired. Ademola *et al.* [5] obtained new results on the stability and boundedness of solutions to a certain second-order nonautonomous stochastic differential equation of the form

$$x''(t) + g_6(x(t),x'(t))x'(t) + f_6(x(t)) + \vartheta x(t)\theta'(t) = p_6(t,x(t),x'(t)).$$

However, our interest is in the study of the asymptotic behaviour of solutions of the differential equations

$$x''(t) + g(t,x(t),x'(t))x'(t) + f(x(t)) = p(t,x(t),x'(t)); \tag{1.1}$$

$$x''(t) + g(t,x(t),x'(t))x'(t) + f(x(t-h)) = p(\cdot); \tag{1.2}$$

and

$$x''(t) + g(t,x(t),x'(t))x'(t) + f(x(t)) + \sigma x(t)\omega'(t) = p(\cdot), \tag{1.3}$$

where σ, h, are positive constants with h being the delay constant, the functions, $g \in C([\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R},\mathbb{R})$, $p(\cdot) = p(t,x(t),$
$x'(t))$, $p \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are continuous functions, $\mathbb{R}^+ = [0,\infty), \mathbb{R} = (-\infty,\infty)$ and $\omega(t) \in \mathbb{R}^m$ (an $m-$dimensional standard Brownian motion (or Wiener process) defined on a probability space). The remaining parts of the paper are as follows, definitions and some basic results on the asymptotic behaviour of solutions of vector ordinary, delay, and stochastic differential equations are given in Section 2. We state and prove the main results of this paper in Section 3, while examples are given in the last section.

2. Preliminaries

Consider the system

$$y' = f(s,y), \tag{2.1}$$

where y is an $n-$vector. Suppose that $f(s,y)$ is continuous on $\mathbb{R}^+ \times D$, where D is a connected open set in $\mathbb{R}^n$. Let C be a class of solutions of equation (2.1) which remain in D, $y_0(s)$ be an element of C and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Definition 2.1. (See [37]) The solution $y(s) \equiv 0$ of equation (2.1) is stable, if for any $\varepsilon > 0$ and any $s_0 \in \mathbb{R}^+$, there exists a $\delta(\varepsilon, s_0) > 0$ such that $\|y(s;s_0,y_0,)\| < \varepsilon$ for all $s \geq s_0$ whenever $\|y_0\| < \delta(\varepsilon, s_0)$.

Definition 2.2. The zero solution of equation (2.1) is asymptotically stable, if $y(s) \equiv 0$ is stable and if there exists a $\delta_0(s_0) > 0$ such that $y(s;s_0,y_0) \to 0$ as $s \to \infty$ whenever $\|y_0\| < \delta_0(s_0)$.

Definition 2.3. A solution $y(s;s_0,y_0)$ of equation (2.1) is bounded, if there exists a $\delta > 0$ such that $\|y(s;s_0,y_0)\| < \delta$, for all $s \geq 0$, where δ may depend on each solution.

Definition 2.4. A solution $y(s;s_0,y_0)$ of equation (2.1) is ultimately bounded for bound E, if there exists a $E > 0$ and a $T_1 > 0$ such that for every solution $y(s;s_0,y_0)$ of equation (2.1), $\|y(s;s_0,y_0)\|$
$< E$, for all $s \geq s_0 + T_1$, where E is independent of the particular solution while T_1 may depend on each solution.

Lemma 2.5. (See [37]). Suppose that there exists a Lyaponuv function $V(s,y)$ defined on $0 \leq s < \infty$, $\|y\| < H$, $H > 0$ which satisfies the following conditions

(i) $V(s,0) \equiv 0$;
(ii) $a(\|y\|) \leq V(s,y)$, where $a(\nu)$ is continuous and increasing; and
(iii) $V'_{(2.1)}(s,y) \leq 0$, (the upper right-hand derivative of the function V along the solution path of (2.1)).

Then the zero solution of the system (2.1) is stable.

Lemma 2.6. (See [37]). Suppose that there exists a Lyaponuv function $V(s,y)$ defined on $0 \leq s < \infty$, $\|y\| < H$, $H > 0$ which satisfies the following conditions

(i) $V(s,0) \equiv 0$;
(ii) $a(\|y\|) \leq V(s,y) \leq b(\|y\|)$, where $a(\nu), b(\nu)$ are continuous increasing and $a(\nu) \to \infty$ as $\nu \to \infty$; and
(iii) $V'_{(2.1)}(s,y) \leq -c(\|y\|)$, where $c(\nu)$ is positive and continuous.

Then the zero solution of the system (2.1) is uniformly asymptotically stable.

Lemma 2.7. (See [37]) Suppose that there exists a non-negative Lyapunov function $V(s,y)$ on $I \times Q$ such that $V'_{(2.1)}(s,y) \leq -W(y)$, where $W(y)$ is positive definite with respect to a closed set Ω in the space Q. Moreover, suppose that $f(s,y)$ of the system (2.1) is bounded for all s when y belongs to an arbitrary compact set in Q and that $f(s,y)$ satisfies conditions

(a) $f(s,y)$ tends to a function $H(y)$ for $y \in Q$ as $s \to \infty$ and on any compact set in Q this convergence is uniform. Consequently, $H(y)$ is a continuous function on Q; and

(b) Corresponding to each $\varepsilon > 0$ and each $y \in Q$, there exists a $\delta(\varepsilon, z) > 0$ and a $T(\varepsilon, z) > 0$ such that if $\|y - z\| < \delta(\varepsilon, z)$ and $s \geq T(\varepsilon, z)$, we have $\|f(s, y) - (s, z)\| < \varepsilon$

with respect to Q. Then, every bounded solution of (2.1) approaches the largest semi-invariant set of the system

$$y' = H(y) \tag{2.2}$$

contained in Ω as $s \to \infty$. In particular, if all solutions of (2.1) are bounded, every solution of (2.1) approaches the largest semi-invariant set of (2.2) contained in Ω as $s \to +\infty$.

Next, for $y \in \mathbb{R}^n$, $\|y\|$ is the norm of y, and a given $r > 0$, C denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\varphi \in C$,

$$\|\varphi\| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|.$$

C_H will denote the set of $\varphi \in C$ such that $\|\varphi\| < H$. For any continuous function $y(u)$ defined on $-r \leq u < A$, $A > 0$, and any fixed s, $0 \leq s < A$, the symbol y_s, will denote the restriction of $y(u)$ to the interval $[s - r, s]$, i.e., y, is an element of C defined by $y_s(\theta) = y(s + \theta)$, $-r \leq \theta \leq 0$. Let $y'(s)$ denote the right-hand derivative of $y(u)$ at $u = s$, and consider the functional-differential equation

$$y'(s) = f(s, y_s), \tag{2.3}$$

where $s \in \mathbb{R}$ and the initial value $\varphi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$, C_H is an open ball of radius H in C;

$$C_H := \{\varphi \in C([-r, 0], \mathbb{R}^n) : \|\varphi\| < H\}.$$

Definition 2.8. (See [37]). A function $y(s_0, \varphi)$ is said to be a solution of equation (2.3) with initial condition $\varphi \in C_H$ at $s = s_0$, $s_0 \geq 0$, if there is a $T > 0$ such that $y(s_0, \varphi)$ is a function from $[s_0 - r, s_0 + T)$ into $\mathbb{R}^n$, with the properties

(i) $y_s(s_0, \varphi) \in \mathbb{R}^n$ for $s_0 \leq s < s_0 + T$;
(ii) $y_s(s_0, \varphi) = \varphi$; and
(iii) $y(s_0, \varphi)$ satisfies equation (2.3) for $s_0 \leq s < s_0 + T$.

The definition of stability and boundedness can be given in the same way as for ordinary differential equations, that is, by replacing the initial value y_0 and the solution $y(s; s_0, y_0)$ by φ and $y_s(s_0, \varphi)$, respectively.

Definition 2.9. (See [37]) A functional $V = V(s, \varphi)$ defined on $s \in \mathbb{R}^+$, $\varphi \in C_H$ is called Lyapunov functional for the system (2.3) if

(i) $a(\|\varphi\|) \leq V(s, \varphi) \leq b(\|\varphi\|)$, where $a(\nu)$ and $b(\nu)$ are continuous increasing and $a(\nu) \to \infty$ as $\nu \to \infty$; and
(ii) $V'_{(2.3)}(s, \varphi) \leq -c(\|\varphi\|)$, where $c(\nu)$ is continuous and positive for $\nu > 0$.

Definition 2.10. Let $V(s, \varphi)$ be a continuous functional defined for $s \geq 0$, $\varphi \in C_H$. The derivative of this functional V along the solutions path of equation (2.3) is defined to be

$$V'_{(2.3)}(s, \varphi) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}[V(s + \delta, y_{s+\delta}(s, \varphi)) - V(s, y_s(s_0, \varphi))],$$

where $y(s_0, \varphi)$ is the solution of equation (2.3) with $y_{s_0}(s_0, \varphi) = \varphi$.

Lemma 2.11. (See [37]). Let $V : \mathbb{R}^+ \times C_H \to \mathbb{R}$ be continuous and locally Lipschitz in φ, $V(s,0) = 0$ and such that

(i) $W_1(|\varphi|) \le V(s,\varphi) \le W_2(|\varphi|)$; and

(ii) $V'_{(2.3)}(s,\varphi) \le 0$, where W_i $(i = 1,2)$ are wedges.

Then the solution y_s of equation (2.3) is uniformly stable. If we defined

$$\Omega = \{\varphi \in C_H : V'_{(2.3)}(\varphi) = 0\},$$

then $y_s = 0$ of equation (2.3) is asymptotically stable, provided that the largest invariant set in $\Omega = \{\mathbf{0}\}$.

Lemma 2.12. (See [37]). Let $V(s,\varphi)$ be a continuous Lyaponuv functional on C and let U_l denote the region such that $V(s,\varphi) < l$. Suppose that $V(s,\varphi) \ge 0$ and $V'_{(2.3)}(s,\varphi) \le 0$ for all $(s,\varphi) \in U_l$ and that there exists a constant $K \ge 0$ such that $|\varphi(s_0,\varphi_0)| \le K$ for all $(s,\varphi) \in U_l$. If E is the set of all points in U_l where $V'_{(2.3)}(s,\varphi) = 0$ and M is the largest invariant set in E, then every solution of equation (2.3) with initial value in U_l approaches M as $s \to \infty$.

Finally, consider the nonautonomous n-dimensional stochastic differential equation.

$$dy_s = F(s,y_s)ds + G(s,y_s)d\Omega_s, \tag{2.4}$$

on $s \ge s_0$ with initial value $y_{s_0} = y_0$, where $F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions, $\Omega_s \in \mathbb{R}^m$ (an $m-$dimensional Wiener process defined on a probability space). Suppose that both F and G are sufficiently smooth for equation (2.4) to have a unique continuous solution on $s \ge 0$ which is denoted by $y(s,y_0)$, if $y(0) = 0$, with further assumption that $F(s,0) = 0, G(s,0) = 0$ for all $s \ge 0$.

Then, the stochastic differential equation (2.4) admits zero solution $y(s,0) \equiv 0$.

Definition 2.13. The zero solution of the stochastic differential equation (2.4) is said to be stochastically stable or stable in probability, if for every pair of $\varepsilon \in (0,1)$ and $\eta > 0$, $\exists$ a $\delta_0 = \delta_0(\varepsilon,\eta) > 0$ such that $P\{|y(s;y_0)| < \eta,\ \forall\ s \ge 0\} \ge 1-\varepsilon$, whenever $|y_0| < \delta_0$.

Definition 2.14. The zero solution of the stochastic differential equation (2.4) is said to be stochastically asymptotically stable, if it is stochastically stable and in addition if for every pair of $\varepsilon \in (0,1)$ and $\eta > 0$, $\exists$ a $\delta_0 = \delta_0(\varepsilon) > 0$ such that $Pr\{\lim_{s\to\infty} y(s;y_0) = 0\} \ge 1-\varepsilon$, whenever $|y_0| < \delta_0$.

Definition 2.15. A solution $y(s,s_0,y_0)$ of the stochastic differential equation (2.4) is said to be stochastically bounded or bounded in probability if it satisfies $E^{y_0}\|y(s,y_0)\| \le C(s_0,\|y_0\|)$, for all $s \ge s_0$, where E^{y_0} denotes the expectation operator with respect to the probability law associated with y_0, $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on s_0 and y_0.

Definition 2.16. The solution $y(s,s_0,y_0)$ of the stochastic differential equation (2.4) is said to be uniformly stochastically bounded, if it satisfies
$E^{y_0}\|y(s,y_0)\| \le C(\|y_0\|)$, for all $s \ge s_0$, where E^{y_0} denotes the expectation operator with respect to the probability law associated with y_0, $C : \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on y_0.

Let $V \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^+ \times \mathbb{R}^n)$ denote the family of nonnegative functions $V(s,y)$ (Lyapunov function) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are once continuously differentiable in s and twice continuously differentiable in y. By Itô's formula, we have

$$dV(s,y) = LV(s,y)ds + V_y(s,y)G(s,y)d\omega(s), \tag{2.5}$$

where

$$\begin{aligned} LV(s,y) = {} & \frac{\partial V(s,y)}{\partial s} + \frac{\partial V(s,y)}{\partial y}F(s,y) \\ & + \frac{1}{2}trace[G^T(s,y)V_{yy}(s,y)G(s,y)], \end{aligned}$$

$$V_y(s,y) = \left(\frac{\partial V(s,y)}{\partial y_1}, \cdots, \frac{\partial V(s,y)}{\partial y_n}\right)$$

and

$$V_{yy}(s,y) = \left(\frac{\partial^2 V(s,y)}{\partial y_i \partial y_j}\right)_{n\times n}, \qquad i,j = 1,2,\cdots,n.$$

Lemma 2.17. (See [31]). Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and a positive constant ϕ such that

(i) $V(s,0) = 0$;
(ii) $V(s,y(s)) > \phi(\|y(s)\|)$; and
(iii) $LV(s,y) \le 0$ for all $(s,y) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic differential equation (2.4) is stochastically stable.

Lemma 2.18. (See [31]). Suppose that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and positive constants ϕ_0, ϕ_1, ϕ_2 such that

(i) $V(s,0) = 0$;
(ii) $\phi_0(\|y\|) \le V(s,y) \le \phi_1(\|y\|), \phi_0(\nu) \to \infty$ as ν tends to infinity; and
(iii) $LV(s,y) \le -\phi_2(\|y\|)$ for all $(s,y) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of stochastic differential equation (2.4) is uniformly stochastically asymptotically stable in the large.

Assumption 2.19. (See [28]). Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, and suppose that for any solutions $y(s_0, y_0)$ of stochastic differential equation (2.4) and for any fixed $0 \le s_0 \le T < \infty$, we have

$$E^{y_0}\left\{\int_{s_0}^T V_{y_i}^2(s,y(s))G_{ik}^2(s,y(s))ds\right\} < \infty, \tag{2.6}$$

where $1 \le i \le n, \quad 1 \le k \le m$.

Assumption 2.20. (See [28]). A special case of the general inequality (2.6) is the following condition. Assume that there exists a function $\rho(s)$ such that

$$|V_{y_i}(s,y(s))G_{ik}(s,y(s))| < \rho(s), \tag{2.7}$$

where $y \in \mathbb{R}^2 \qquad 1 \le i \le n, \quad 1 \le k \le m$, and for any fixed $0 \le s_0 \le T < \infty$,

$$\int_{s_0}^T \rho^2(s)ds < \infty. \tag{2.8}$$

Lemma 2.21. (See [28]). Assume that there exists a Lyapunov function $V(s,y(s)) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, satisfying Assumption 2.19, such that, for all $(s,y(s)) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|y(s)\|^j \leq V(s,y(s))$
(ii) $LV(s,y(s)) \leq -\alpha(s)\|y(s)\|^k + \psi(s)$; and
(iii) $V(s,y(s)) - V^k(s,y(s)) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+;\mathbb{R}^+)$, j,k and μ are positive constants, $j \geq 1$ and ψ is a non negative constant. Then the solutions of the stochastic differential equation (2.4) satisfy

$$E^{y_0}\|y(s,y_0)\| \leq \Big[V(s_0,y_0)e^{-\int_{s_0}^{s}\alpha(\varepsilon)d\varepsilon} + \int_{s_0}^{s}(\mu\alpha(u) + \psi(u))e^{-\int_{u}^{s}\alpha(\varepsilon)d\varepsilon}du\Big]^{1/j}, \tag{2.9}$$

for all $s \geq s_0$.

Lemma 2.22. (See [28]). Assume there exist a Lyapunov function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, satisfying Assumption 2.20, such that for all $(s,y(s)) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|y(s)\|^j \leq V(s,y(s)) \leq \|y(s)\|^k$;
(ii) $LV(s,y(s)) \leq -\alpha(s)\|y(s)\|^\nu + \psi(s)$; and
(iii) $V(s,y(s)) - V^{\nu/k}(s,y(s)) \leq \mu$,

where $\alpha, \psi \in C(\mathbb{R}^+;\mathbb{R}^+)$, j,k and ν are positive constants, $j \geq 1$ and ψ is a non negative constant. Then the solutions of the stochastic differential equation (2.4) satisfy inequality (2.9) for all $s \geq s_0$.

Corollary 2.23. (See [28]).

(a) Suppose all the hypotheses of Lemma 2.21 hold and in addition,

$$\int_{s_0}^{s}(\mu\alpha(u) + \psi(u))e^{-\int_{u}^{s}\alpha(\varepsilon)d\varepsilon}du \leq B, \tag{2.10}$$

for all $s \geq s_0 \geq 0$, for some positive constant B; then all solutions of stochastic differential equation (2.4) are stochastically bounded.

(b) Again suppose all the hypotheses of Lemma 2.22 hold and in addition, if condition (2.10) is satisfied; then all solutions of stochastic differential equation (2.4) are uniformly stochastically bounded.

3. Main Results

This section presents boundedness and the behaviour of solutions as $t \to \infty$. The equivalent system of equation (1.1) is

$$x' = y,\ y' = p(t,x,y) - g(t,x,y)y - f(x) \tag{3.1}$$

where the functions f, g and p are defined in Section 1. We define the Lyapunov function $V(x,y) = V$ as

$$2V = (a^2+b^2)x^2 + (b+1)y^2 + 2axy + xf(x), \tag{3.2}$$

for all $a > 0$, and $b > 0$ are constants.

Theorem 3.1. Further to the basic assumption on the functions g, f and p in equation (1.1), suppose that a, b, b_1, c, β are positive constants and that

(i) $g(t,x,y) \geq a$;
(ii) $f(0)=0$, $b \leq \frac{f(x)}{x} \leq b_1$, for all $x \neq 0$, $f'(x) \leq c$ for all x, $b > \dfrac{c}{2a}$; and
(iii) $|p(t,x,y)| \leq \beta$ for all $t \geq 0$.

Then every solution of system (3.1) satisfies

$$x^2(t)+y^2(t) \leq \exp(-\eta t) + \left\{P_1 + P_2 \int_{t_0}^{t} |p(s,x,y)| \exp\left(\frac{\eta s}{2}\right) ds\right\}^2,$$

for all $t \geq t_0$, where the constant $P_1 = P_1(\lambda_0, \lambda_2, \lambda_3, \lambda_7, a, b, x(t_0,), y(t_0), \delta_1, \delta_2,) > 0$ and the constant $P_2 = P_2(\lambda_1, \lambda_3, \lambda_4, \lambda_5) > 0$.

Theorem 3.2. Suppose that conditions (i) to (iii) of Theorem 3.1 hold, and in addition $\int_0^\infty |p(t,x,y)|dt < \beta < \infty$, for all $t \geq 0$, x and y. Then every solution $(x(t),y(t))$ of the system (3.1) satisfies

$$\lim_{t\to\infty} x(t) = 0; \qquad \lim_{t\to\infty} y(t) = 0. \tag{3.3}$$

One influential lemma that is extremely important to the proofs of Theorems 3.1 and 3.2 will be stated and proved.

Lemma 3.3. Suppose all the conditions of Theorem 3.1 hold, then there exist positive constants $\lambda_1 = \lambda_1(b)$ and $\lambda_2 = \lambda_2(a,b,b_1)$ such that

$$\lambda_1(x^2+y^2) \leq V \leq \lambda_2(x^2+y^2) \tag{3.4}$$

for all x and y. Furthermore, there exist positive constants $\lambda_3 = \lambda_3(\delta_1, \delta_2)$ and $\lambda_4 = \lambda_4(a,b)$ such that along the solution path of system (3.1)

$$V'_{(3.1)} \leq -\lambda_3(x^2+y^2) + \lambda_4(x^2+y^2)^{1/2}|p(t,x,y)| \ \forall\, x \text{ and } y. \tag{3.5}$$

Proof. Let $(x(t),y(t))$ be any solution of (3.1). From equation (3.2), it is clear that $V(0,0)=0$,for all $t \geq 0$. Re-writing equation (3.2), we have

$$V = \frac{1}{2}\left\{(ax+y)^2 + \left(b^2 + \frac{f(x)}{x}\right)x^2 + by^2\right\}.$$

Since $(ax+y)^2 \geq 0$ for all x and y, and from hypothesis (ii) of Theorem 3.1, there exists a positive constant d_1 such that

$$V \geq d_1(x^2+y^2) \tag{3.6}$$

with $d_1 = \frac{1}{2}\min\{b^2+b, b\}$. Applying the inequality $|xy| \leq \frac{1}{2}(x^2+y^2)$ on equation (3.2) gives

$$V \leq \frac{1}{2}\left(\left[a^2 + \left(b^2+b_1\right) + a\right]x^2 + (b+1+a)y^2\right).$$

There exists a positive constant d_2 such that

$$V \leq d_2(x^2+y^2), \tag{3.7}$$

where $d_2 := \frac{1}{2}\max\{a^2+b^2+a+b_1, 1+a+b\}$. From inequalities (3.6) and (3.7), inequality (3.4) is established with d_1 and d_2 equivalent to λ_1 and λ_2 respectively.

Moreover, the time derivative of equation (3.2) along the system (3.1) gives

$$\begin{aligned} V_{(3.1)} = -a\frac{f(x)}{x}x^2 - \Big[(b+1)g(t,x,y) - a\Big]y^2 - \Big[ag(t,x,y) \\ + b\frac{f(x)}{x} - f'(x) - (a^2+b^2)\Big]xy + \Big[ax + (b+1)y\Big]p(t,x,y). \end{aligned} \tag{3.8}$$

Following the hypotheses (i) and (ii) of Theorem 3.1, equation (3.8) could be simplified as

$$V'_{(3.1)} \leq -abx^2 - aby^2 + c|xy| + \Big[a|x| + (b+1)|y|\Big]|p(t,x,y)|. \tag{3.9}$$

Applying the inequality $|xy| \leq \frac{1}{2}(x^2+y^2)$ and the fact that $(|x|+|y|)^2 \leq 2(x^2+y^2)$ for all $x,y \in \mathbb{R}$, in inequality (3.9), there exist two positive constants d_3 and d_4 such that

$$V'_{(3.1)} \leq -d_3(x^2+y^2) + d_4(x^2+y^2)^{1/2}|p(t,x,y)| \tag{3.10}$$

with $d_3 := ab - \frac{1}{2}c$ and $d_4 := \sqrt{2}\max\{a, b+1\}$. Inequality (3.10) establishes inequality (3.5) with d_3 and d_4 equivalent to λ_3 and λ_4 respectively. □

Remarks 1. If $p(t,x,y) \equiv 0$, inequality (3.10) becomes

$$V'_{(3.1)} \leq -\lambda_3(x^2+y^2) \leq 0. \tag{3.11}$$

Proof of Theorem 3.1. Let $(x(t), y(t))$ be any solution of the system (3.1). Recall from equation (3.5) that $V'_{(3.1)} \leq -\lambda_3(x^2+y^2) + \lambda_4(x^2+y^2)^{1/2}|p(t,x,y)|$, for all $t \geq 0$, x and y. From inequalities (3.6) and (3.7), we have

$$(x^2+y^2)^{1/2} \leq \Big(\frac{V}{d_1}\Big)^{1/2} \text{and } -\lambda_3(x^2+y^2) \leq -\lambda_3\Big(\frac{V}{d_2}\Big), \tag{3.12}$$

where λ_3 is a positive constant. Using inequalities (3.12) in (3.5), we have

$$V'_{(3.1)} \leq -\lambda_5 V + \lambda_6 V^{1/2}|p(t,x,y)|, \tag{3.13}$$

where $\lambda_5 := \dfrac{\lambda_3}{d_2} > 0$ and $\lambda_6 := \dfrac{\lambda_4}{(d_1)^{1/2}} > 0$. Inequality (3.13) can be further written as

$$V'_{(3.1)} \leq -2\lambda_7 V + \lambda_6 V^{1/2}|p(t,x,y)|, \tag{3.14}$$

where $\lambda_7 := \dfrac{\lambda_5}{2}$. Further simplification of inequality (3.14) gives

$$V'_{(3.1)} + \lambda_7 V \leq \lambda_6 V^{1/2}\Big[|p(t,x,y)| - \lambda_8 V^{1/2}\Big], \tag{3.15}$$

where $\lambda_8 := \dfrac{\lambda_7}{\lambda_6} > 0$. Inequality (3.15) can be written as

$$V'_{(3.1)} + \lambda_7 V \leq \lambda_6 V^{1/2}|p(t,x,y)|. \tag{3.16}$$

Dividing inequality (3.16) by $V^{1/2}$ and from hypothesis (iii) of Theorem 3.1, we have first order differential inequality

$$V'_{(3.1)}V^{-1/2} + \lambda_7 V^{1/2} \leq \lambda_6 \beta. \tag{3.17}$$

On multiplying inequality (3.17) by $e^{\frac{\lambda_7}{2}t}$ and integrating from t_0 to t, we have

$$V^{1/2}(t) \le e^{-\frac{\lambda_7}{2}t}\Big[V^{1/2}(t_0)e^{\frac{\lambda_7}{2}t_0} + \frac{\lambda_6}{2}\int_{t_0}^{t}\beta e^{\frac{\lambda_7}{2}s}ds\Big]. \tag{3.18}$$

Using inequality (3.4) and (3.18), we obtain

$$x^2(t)+y^2(t) \le e^{-\lambda_7 t}\Big[P_1 + P_2\int_{t_0}^{t} e^{\frac{\lambda_7}{2}s}ds\Big]^2,$$

where $P_1 = \frac{\lambda_2}{\lambda_1}\Big(x^2(t_0)+y^2(t_0)\Big)^{1/2}e^{\frac{\lambda_7}{2}t_0}$ and $P_2 = \frac{\lambda_6\beta}{2\lambda_1}$. For convenience, let $\lambda_7 = \eta$, then we have

$$x^2(t)+y^2(t) \le e^{-\eta t}\Big[P_1 + P_2\int_{t_0}^{t} e^{\frac{\eta s}{2}}ds\Big]^2. \tag{3.19}$$

□

Remarks 2. If $p(t,x,y)=0$, then inequality (3.19) reduces to $x^2(t)+y^2(t) \le \lambda_9 e^{-\eta t}P_1$ and $x^2(t)+y^2(t) \to 0$, as $t\to\infty$, which established that, the zero solution of the system (3.1) is asymptotically stable.

Proof of Theorem 3.2. Let $(x(t),y(t))$ be any solution of the system (3.1). Now from inequality (3.5) and choosing $(x^2+y^2)^{1/2} \ge \lambda_{10}$ where $\lambda_{10} := 2\beta\lambda_3^{-1}\lambda_4 > 0$ is a constant, then inequality (3.5) becomes

$$V'_{(3.1)} \le -\frac{1}{2}\lambda_3(x^2+y^2) \le 0, \text{ for all } x,y \text{ and } t\ge 0. \tag{3.20}$$

Consider the set $\Omega_1 = \{X(t) = (x(t),y(t)) \in \mathbb{R}^2 | V'_{(3.1)}(t,X(t)) = 0\}$. Since equation (1.1) can be written in the form $X' = F(X(t)) + G(t,X(t))$, where $F(X(t)) = (y, -g(t,x,y)y - f(x))^T$ and $G(t,X(t))$
$= (0, p(t,x,y))^T$. Then by inequality (3.20), and the fact that
$V'_{(3.1)}(X(t)) = 0$ on Ω_1 implies that $x=y=0$, it follows that $X' = \mathbf{0}$ has solution $(X(t))^T = K^T$, where $X(t) = (x(t),y(t)) \in \mathbb{R}^2$ and $K = (k_1,k_2)$. For $X(t)\in\mathbb{R}^2$ to remain in Ω_1, we must have $k_1=k_2=0$. The largest invariant set in Ω_1 is $\{0,0\}$ so that by inequality (3.6), (3.7) and (3.20), all assumptions of Lemma 2.7 holds with $\varphi \equiv X(t) \in \mathbb{R}^2$, hence by Lemma 2.7, equation (3.3) is established. Hence the proof. □

Now we consider equation (1.2) and its equivalent system

$$\begin{aligned} x' &= y,\\ y' &= p(t,x,y) - g(t,x,y)y - f(x) + \int_{t-h}^{t} f'(x(s))y(s)ds. \end{aligned} \tag{3.21}$$

where $0<h$, is a constant to be determined. The main tool to be used in proving our results is the continuously differentiable Lyapunov functional $V = V(t,x_t,y_t)$ defined as

$$\begin{aligned} 2V &= b^2x^2 + by^2 + (ax+y)^2 + 2\int_0^x f(s)ds\\ &\quad + \gamma\int_{-h}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds, \end{aligned} \tag{3.22}$$

where γ is a positive constant that will be determined later.

Theorem 3.4. In addition to the basic assumption on the functions f, g and p in equation (1.2), suppose that a, b, h, β, P, Q are positive constants such that

(i) $g(t,x,y) \geq a$ for all x and y;
(ii) $f(0) = 0, b \leq \frac{f(x)}{x} \leq \beta$ for all $x \neq 0$;
(iii) $|f'(x)| \leq P$ for all x;
(iv) $0 < h$ where

$$h < \min\left\{\frac{ab}{P(1+2a+b)}, \frac{ab}{P(2+a+2b)}\right\}; \text{and} \tag{3.23}$$

(v) $|p(t,x,y)| \leq Q, \quad Q < \infty$.

Then every solution (x_t, y_t) of the system (3.21) satisfies

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} y(t) = 0. \tag{3.24}$$

A lemma will be stated and proved which will be used in the proofs of our results.

Lemma 3.5. Suppose all the conditions of Theorem 3.4 holds, then there exist positive constants $\lambda_{11}, \lambda_{12}, \lambda_{13}$ such that

$$\lambda_{11}(x^2+y^2) \leq V \leq \lambda_{12}(x^2+y^2) + \lambda_{13}\int_{t-h}^{t} y^2(s)ds \tag{3.25}$$

for all $t \geq 0$, x and y. Moreover, there exist positive constants λ_{14} and λ_{15} such that along the solution path of system (3.21)

$$V'_{(3.21)} \leq -\lambda_{14}(x^2+y^2) + \lambda_{15}(|x|+|y|)|p(t,x,y)|. \tag{3.26}$$

Proof. Let (x_t, y_t) be any solution of (3.21). From equation (3.22) we have $V(t,0,0) = 0$ for all $t \geq 0$. Also, the double integrals term in (3.22) is nonnegative, $\frac{f(x)}{x} \geq b$ for all $x \neq 0$, and $(ax+y)^2 \geq 0$ for all x and y, there exists a positive constant $\lambda_{11} = \lambda_{11}(b)$ such that

$$V \geq \lambda_{11}(x^2+y^2), \tag{3.27}$$

where $\lambda_{11} := \frac{b}{2}\min\{1+b, 1\}$. Applying the inequality $|xy| \leq \frac{1}{2}(x^2+y^2)$ on equation (3.22), since $\frac{f(x)}{x} \leq \beta$ for all $x \neq 0$, there exist positive constants λ_{12} and λ_{13} such that

$$V \leq \lambda_{12}(x^2+y^2) + \lambda_{13}\int_{t-h}^{t} y^2(s)ds, \tag{3.28}$$

where $\lambda_{12} = \frac{1}{2}\max\{(a^2+b^2+a+\beta), (a+b+1)\}$, and $\lambda_{13} = \frac{1}{2}\gamma$. Hence, from inequality (3.27) and (3.28), inequality (3.25) is established.

Furthermore, the time derivative of the functional V defined by equation (3.22) along the solution path of the system (3.21) is

$$\begin{aligned} V'_{(3.21)} &= -a\frac{f(x)}{x}x^2 - \Big[(b+1)g(t,x,y) - a\Big]y^2 \\ &- \Big[b\frac{f(x)}{x} + ag(t,x,y) - (a^2+b^2)\Big]xy + \Big[ax + (b+1)y\Big]\times \\ &p(t,x,y) + \Big[ax+(b+1)y\Big]\int_{t-h}^{t} f'(x(s))y(s)ds \\ &+ \frac{\gamma}{2}\Big[hy^2 - \int_{t-h}^{t} y^2(s)ds\Big]. \end{aligned} \tag{3.29}$$

Applying hypotheses (i)-(iii) of Theorem 3.4, and the obvious inequality $|xy| \le \frac{1}{2}(x^2 + y^2)$ in equation (3.29), we have

$$\begin{aligned} V'_{(3.21)} &\le -\frac{1}{2}abx^2 - \frac{1}{2}aby^2 + \frac{P}{2}\Big[ax^2 + (b+1)y^2\Big]h + \frac{\gamma}{2}\Big[hy^2 \\ &- \int_{t-h}^{t} y^2(s)ds\Big] + \frac{P}{2}(a+b+1)\int_{t-h}^{t} y^2(s)ds \\ &- (W_5 + W_6) + \max\{a, b+1\}(|x|+|y|)|p(t,x,y)|, \end{aligned} \tag{3.30}$$

where $W_5 := \frac{1}{4}\Big[abx^2 + 4(ag(t,x,y) - a^2)xy + aby^2\Big]$ and $W_6 := \frac{1}{4}\Big[abx^2 + 4(b\frac{f(x)}{x} - b^2)xy + aby^2\Big]$. Since W_5 and W_6, are quadratic expressions, it is not difficult to show that

$$W_5 = W_6 \ge \frac{1}{4}ab\Big[|x| - |y|\Big]^2 \ge 0, \tag{3.31}$$

for all x and y. Using estimate (3.31) in (3.30) we have

$$\begin{aligned} V'_{(3.21)} &\le -\frac{1}{2}abx^2 - \frac{1}{2}aby^2 + \frac{h}{2}\Big[(b+1)P + \gamma\Big]y^2 \\ &+ \frac{1}{2}\Big[\gamma - (a+b+1)P\Big]\int_{t-h}^{t} y^2(s)ds + \frac{h}{2}(aP+\gamma)x^2 \\ &+ \max\{a, b+1\}(|x|+|y|)|p(t,x,y)|. \end{aligned} \tag{3.32}$$

On choosing $\gamma = (a+b+1)P > 0$, there exist positive constants λ_{14} and λ_{15} such that

$$V' \le -\lambda_{14}(x^2+y^2) + \lambda_{15}(|x|+|y|)|p(t,x,y)| \tag{3.33}$$

for all x and $y \neq 0$, where $\lambda_{14} := \frac{1}{2}\min\Big\{ab - (2a+b+1)hP,\ ab - (a+2b+2)hP\Big\}$ and $\lambda_{15} := \max\{a, b+1\}$. $\square$

Proof of Theorem 3.4. Let (x_t, y_t) be any solution of the system (3.21). Since inequality $(|x|+|y|)^2 \le 2(x^2+y^2)$, then inequality (3.33) can be written as

$$V' \le -\lambda_{14}(x^2+y^2) + \lambda_{16}(x^2+y^2)^{1/2}, \tag{3.34}$$

with $\lambda_{16} = \sqrt{2}\lambda_{15}Q$. There exist positive constants λ_{17} and λ_{18} such that inequality (3.34) becomes

$$V' \le -\lambda_{17}(x^2+y^2) \le 0, \tag{3.35}$$

for all $t \geq 0$, x and y provided that $(x^2+y^2)^{1/2} \geq \lambda_{18}$, where $\lambda_{17} := \frac{1}{2}\lambda_{14}$ and $\lambda_{18} := 2\lambda_{14}^{-1}\lambda_{16}$. Now consider the set $\Omega_2 = \left\{X_t = (x_t, y_t) \in \mathbb{R}^2 : V'_{(3.21)}(t, X_t) = 0\right\}$. Since the system (3.21) can be written as

$$X' = F(X_t) + G(t, X_t),$$

where

$$F(X_t) = \left(y, -g(t,x,y)y - f(x) + \int_{t-h}^{t} f'(x(s))y(s)ds\right)^T,$$

and

$$G(t, X_t) = (0, p(t,x,y))^T.$$

From inequality (3.35), and the fact that $V'_{(3.21)}(t, X_t) = 0$ on Ω_2, implies that $x = y = 0$ and since $f(0) = 0$, it follows that $X' = \mathbf{0}$ has solution $(x_t, y_t)^T = (k_1, k_2)^T$. For $X_t \in \mathbb{R}^2$ to remain in Ω_2, we must have $k_1 = k_2 = 0$. Since the largest invariant set in Ω_2 is $\{0,0\}$ so that by inequality (3.27), (3.28) and (3.35), all assumptions of Lemma 2.12 holds and hence by Lemma 2.12 equation (3.24) is established. □

Finally, we shift to stochastic differential equation (1.3) and consider the equivalent system

$$x' = y,\ y' = p(t,x,y) - g(t,x,y)y - f(x) - \sigma x\omega'(t), \tag{3.36}$$

where the functions f, g and p are defined in Section 1. The continuously differentiable Lyapunov function $V = V(x(t), y(t))$ employed is defined (3.2) for all $a > 0, b > 0$ constants.

Theorem 3.6. Suppose that a, b, σ, β, and B_0 are positive constants such that

(i) $a \leq g(t,x,y)$ for all x and y, $g(t,0,0) = 0$;
(ii) $f(0) = 0, b \leq \frac{f(x)}{x} \leq \beta$ for all $x \neq 0$;
(iii) $\frac{1}{2}\sigma^2(b+1) < ab$; and
(iv) $|p(t,x,y)| < B_0$ for all $t \geq 0$, x and y.

Then the solution $(x(t), y(t))$ of the stochastic differential system (3.36) is uniformly stochastically bounded.

Theorem 3.7. Suppose that all hypotheses of Theorem 3.6 hold. In addition, if hypotheses $\int_0^\infty |p(t,x,y)|dt < \infty$ is satisfied, for all $t \geq 0$, x and y. Then every solution $(x(t), y(t))$ of the system (3.36) satisfies

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} y(t) = 0. \tag{3.37}$$

The proof of the following lemma will be used in the proofs of Theorems 3.6 and 3.7.

Lemma 3.8. Suppose all the conditions of Theorem 3.6 holds, then there exist positive constants λ_{19} and λ_{20} such that along the solution path of (3.36)

$$\lambda_{19}(x^2+y^2) \leq V \leq \lambda_{20}(x^2+y^2) \tag{3.38}$$

for all x and y. In addition, there exists positive constants λ_{21} and λ_{22} such that

$$LV_{(3.36)}(x,y) \leq -\lambda_{21}(x^2+y^2) + \lambda_{22}(|x|+|y|)|p(t,x,y)| \tag{3.39}$$

for all $t \geq 0, x$ and y.

Proof. See the proof in Lemma (3.3) which establishes estimate (3.38) with d_1 and d_2 equivalent to λ_{19} and λ_{20} respectively. Furthermore, from equation (2.5), we find that

$$\begin{aligned} LV_{(3.36)} = &-\frac{1}{2}\Big[a\frac{f(x)}{x} - \frac{1}{2}\sigma^2(b+1)\Big]x^2 \\ &-\frac{1}{2}\Big[(b+1)g(t,x,y) - a\Big]y^2 + [ax + (b+1)y]p(\cdot) - W_7, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} W_7 := &\frac{1}{2}\Big\{\Big[a\frac{f(x)}{x} - \frac{1}{2}\sigma^2(b+1)\Big]x^2 + 2\Big[b\frac{f(x)}{x} + ag(t,x,y) \\ &-(a^2+b^2+f'(x))\Big]xy + \Big[(b+1)g(t,x,y) - a\Big]y^2\Big\}. \end{aligned}$$

Noting that W_7 is quadratic, it is not difficult to show that

$$\begin{aligned} W_7 \geq &\frac{1}{2}\Bigg[\sqrt{\Big[a\frac{f(x)}{x} - \frac{1}{2}\sigma^2(b+1)\Big]}|x| \\ &-\sqrt{\Big[(b+1)g(t,x,y) - a\Big]}|y|\Bigg]^2 \geq 0, \end{aligned} \tag{3.41}$$

for all x and y. Using estimate (3.41), hypotheses (i) and (ii) of Theorem 3.6 in equation (3.40), we have

$$LV_{(3.36)} \leq -\frac{1}{2}\Big[ab - \frac{1}{2}\sigma^2(b+1)\Big]x^2 - \frac{1}{2}aby^2 + [ax + (b+1)y]p(t,x,y).$$

Using the hypotheses of Theorem 3.6 there exist positive constants d_7 and d_8 such that

$$LV_{(3.36)} \leq -d_7(x^2+y^2) + d_8(|x|+|y|)|p(t,x,y)| \tag{3.42}$$

for all $t \geq 0$, x and y, where $d_7 := \frac{1}{2}\max\Big\{ab - \frac{1}{2}\sigma^2(b+1), ab\Big\}$ and $d_8 := \max\Big\{a, b+1\Big\}$. Hence, inequality (3.42) establishes inequality (3.39) with $d_7 \equiv \lambda_{21}$ and $d_8 \equiv \lambda_{22}$ respectively. □

Proof of Theorem 3.6. Let $(x(t), y(t))$ be any solution of the system (3.36). Re-writing inequality (3.39), noting hypotheses (iv) of Theorem 3.6, we have

$$\begin{aligned} LV_{(3.36)} \leq &-\frac{1}{2}\lambda_{21}(x^2+y^2) - \frac{1}{2}\lambda_{21}\Big[(|x| - B_0\lambda_{21}^{-1}\lambda_{22})^2 \\ &+ (|y| - B_0\lambda_{21}^{-1}\lambda_{22})^2\Big] + \lambda_{24} \end{aligned}$$

for $t \geq 0, x$ and y where $\lambda_{24} := B_0^2\lambda_{21}^{-1}\lambda_{22}^2$. Since $\lambda_{21}, \lambda_{22}$ and B_0 are positive constants and of course $(|x| - B_0\lambda_{21}^{-1}\lambda_{22})^2 + (|y| - B_0\lambda_{21}^{-1}\lambda_{22})^2 \geq 0$, for all x and y, then there exists positive constants λ_{23} such that

$$LV_{(3.36)}(x,y) \leq -\lambda_{23}(x^2+y^2) + \lambda_{24}, \tag{3.43}$$

for $t \geq 0, x$ and y, where $\lambda_{23} = \frac{1}{2}\lambda_{21}$. Therefore, condition (ii) of Lemma 2.22 is satisfied with $\alpha(t) = \lambda_{23}, \quad \psi(t) = \lambda_{24}, \quad r = 2$. Similarly, from inequality (3.6), hypotheses (i) and (iii) of Lemma 2.22 hold with $j = k = 2$ so that $\mu = 0$. Also, from inequality (2.10) of Corollary 2.23, for $\mu = 0$

$$\begin{aligned} \int_{t_0}^{t} \left[(\mu\alpha(u) + \psi(u))e^{-\int_u^t \alpha(s)ds}\right]du &= \lambda_{23}^{-1}\lambda_{24}\left[1 - e^{-\lambda_{23}(t-t_0)}\right] \\ &\leq \lambda_{23}^{-1}\lambda_{24} \end{aligned} \tag{3.44}$$

for $t \geq t_0 \geq 0$. Inequality (3.44) satisfies (2.10) of Corollary 2.23 with $B = \lambda_{24}\lambda_{23}^{-1} > 0$. Furthermore, from equation (2.6) and (3.2) there exits a positive constant λ_{25} such that

$$|V_{y_i}(t,x,y)G_{ik}(t,x,y)| = |\sigma x||[ax + (b+1)y]| \leq \lambda_{25}(x^2 + y^2), \tag{3.45}$$

where $\lambda_{25} := \frac{1}{2}\sigma \max\{2a + b + 1, b + 1\}$. Similarly from estimate (2.8) of Assumption 2.20 and inequality (3.45), we have

$$\lambda_{25}^2 \int_{t_0}^{T} \left(x^2(t) + y^2(t)\right)^2 dt < \infty \tag{3.46}$$

for any fixed $0 \leq t_0 \leq T < \infty$. In conclusion, using inequality (3.45) in (2.9), we have

$$E^{X_0}\|X(t,X_0)\| \leq \left[\lambda_{26}(x_0^2 + y_0^2) + \lambda_{24}\lambda_{23}^{-1}\right]^{1/2} \tag{3.47}$$

for all $t \geq t_0$. All hypotheses of Lemma 2.22 are satisfied and inequality (2.10) holds so that hypotheses of Corollary 2.23 (b) hold, by Corollary 2.23 (b) all solutions $(x(t), y(t))$ of the system (3.36) are uniformly bounded. □

Proof of Theorem 3.7. Let $(x(t), y(t))$ be any solution of the system (3.36). Since $(|x| + |y|) \leq \sqrt{2}(x^2 + y^2)^{1/2}$ and choosing $(x^2 + y^2)^{1/2} \geq d_9$ where $d_9 = 2^{3/2}d_7^{-1}d_8B_0 > 0$ is a constant, then inequality (3.42) becomes

$$LV_{(3.36)}(x,y) \leq -d_{10}(x^2 + y^2) \leq 0 \tag{3.48}$$

for all $t \geq 0$, x and y where $d_{10} := \dfrac{1}{2}d_7$. Now, from the proof of Theorem 3.6 the uniform boundedness of solutions of system (3.36) has been established.

Consider the set $\Omega_3 := \{X(t) = (x(t), y(t)) \in \mathbb{R}^2 | LV_{(3.36)} = 0\}$, and since system (3.36) can be written in the form

$$X'(t) = A(t)X(t) + X(t)\omega'(t) + G(t, X(t)),$$

where

$$X(t) := \begin{pmatrix} x \\ y \end{pmatrix},\ A(t) := \begin{pmatrix} 0 & 1 \\ -\dfrac{f(x)}{x} & -g(t,x,y) \end{pmatrix},\ \omega := \begin{pmatrix} 0 & 0 \\ -\sigma & 0 \end{pmatrix}$$

and

$$G(t, X(t)) := (0, p(t,x,y))^T.$$

By inequality (3.48), and the fact that $LV_{(3.36)}(X(t)) = 0$ on Ω_3 implies that, $x = y = 0$, since $f(0) = 0$, $g(t,0,0) = 0$, it follows that $A(t)X(t) = 0$, $X(t)\omega'(t) = 0$, so that $X' = \mathbf{0}$ has solution $X(t)^T = K^T$, where $X(t) = (x(t), y(t)) \in \mathbb{R}^2$ and $K = (k_1, k_2)$. For $X(t) \in \mathbb{R}^2$ to remain in Ω_3, we must have $k_1 = k_2 = 0$. The largest invariant set of course in Ω_3 is $\{0,0\}$ so that by inequality (3.6), (3.7) and (3.48), all assumptions of Lemma 2.12 hold true and hence estimate (3.37) is established, which complete the proof of Theorem 3.7. □

4. EXAMPLES

In this section, we shall present examples of ordinary, delay, and stochastic differential equations to illustrate applications of the results obtained in the previous section.

Example 4.1. Consider the second-order nonlinear non-autonomous ordinary differential equation

$$x'' + (3+2t+|\cos(xx')|)x' + (x+\sin x) = [1+t+xx']^{-1}. \tag{4.1}$$

Equation (4.1) is equivalent to system

$$\begin{aligned} x' &= y, \\ y' &= (1+t+xy)^{-1} - (x+\sin x) - (3+2t+|\cos(xy)|)y. \end{aligned} \tag{4.2}$$

Compare systems (3.1) and (4.2), the following relations hold:

(i) The function $g(t,x,x') := 3+2t+|\cos(xy)|$. And since $|\cos(xy)| \geq 0$ for all x and y, it follows from hypothesis (i) of Theorem 3.1 that

$$g(t,x,y) = 3+2t+|\cos(xy)| \geq a = 3 \ \forall \ t \geq 0.$$

(ii) The function $f(x) := x+\sin x$, clearly $f(0) = 0$. From hypothesis (ii) of Theorem 3.1 $\dfrac{f(x)}{x} = 1+\dfrac{\sin x}{x}$, $x \neq 0$. Since $-0.2 \leq \dfrac{\sin x}{x} \leq 1$ for all $x \neq 0$ it follows that

$$0.8 = b \leq \frac{f(x)}{x} = 1+\frac{\sin(x)}{x} \leq b_1 = 2, \ x \neq 0.$$

The behaviour of function $f(x)$, $\dfrac{f(x)}{x}$ and the bounds on $\dfrac{f(x)}{x}$ are shown in Figure 1. Again $f'(x) = 1+\cos x \leq c = 2$ for all x. Also, from items (i) and (ii) we find that $b > \dfrac{c}{2a}$ implies that $1 > \dfrac{1}{3}$. Figure 2 shows the paths of $\cos x$, $f'(x)$ and the upper bound of the function $f'(x)$.

(iii) The function $p(t,x,y) := \dfrac{1}{1+t+xy}$. And of course since $1+t+|xy| \geq 1$, $|p(t,x,y)| \leq \beta = 1$, for all $t \geq 0, x$ and y. Hence, since all the hypotheses of Theorem 3.1 are satisfied, then the solution $(x(t),y(t))$ of system (4.2) is bounded.

(iv) In addition, the function $\int_0^\infty \dfrac{1}{1+t+|xy|} dt < \beta < \infty$ for all $t \geq 0$, x and y.

Then every solution $(x(t),y(t))$ of the system (4.2) satisfies

$$\lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} y(t) = 0.$$

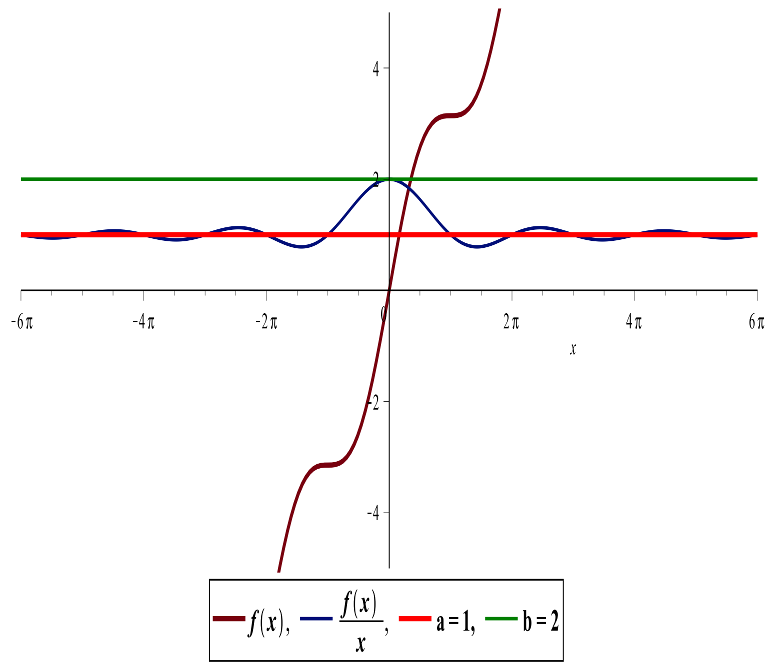

FIGURE 1. Paths and behaviour of $f(x)$, $\dfrac{f(x)}{x}$ and the bounds on $\dfrac{f(x)}{x}$

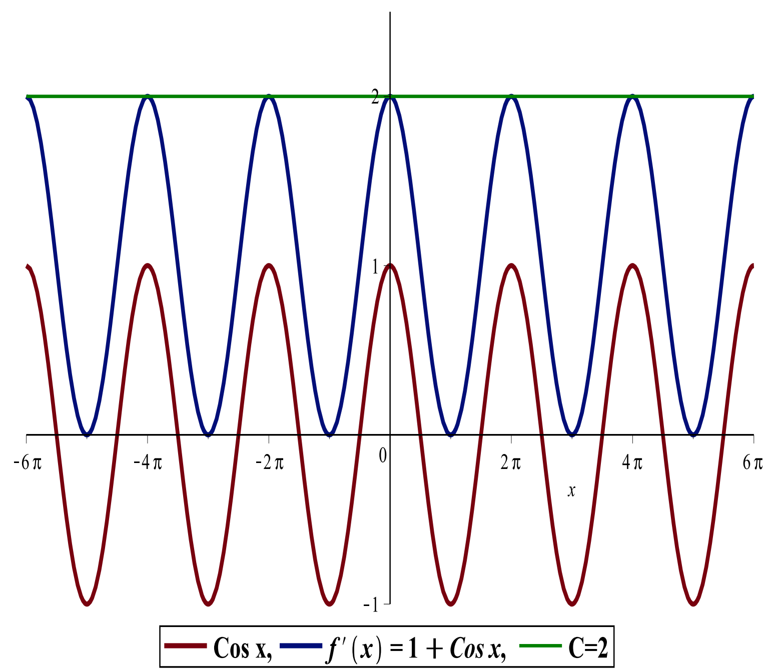

FIGURE 2. Behaviour of $\cos x$ and $f'(x)$ on $[-6\pi, 6\pi]$.

Example 4.2. Suppose $f(x(t)) = f(x(t-h))$ in equation (4.1), we have the following second-order nonlinear non-autonomous delay differential equation

$$x'' + (3+2t+|\cos(xx')|)x' + [(x-h)+\sin(x-h)] = (1+t+xx')^{-1}. \tag{4.3}$$

Equation (4.3) is equivalent to a system of first-order delay differential equation

$$\begin{aligned} x' &= y,\ y' = (1+t+xy)^{-1} - (3+2t+|\cos(xy)|)y \\ &- (x+\sin x) + \int_{t-h}^{t} (y(s)+\cos(s))y(s)ds \end{aligned} \tag{4.4}$$

Recall our considered second-order DDE

$$x''(t) + g(t,x(t),x'(t))x'(t) + f(x(t-h)) = p(t,x(t),x'(t)).$$

As in Example 4.1, the functions g, f and p of equation (4.4) satisfy conditions of Theorem 3.4. Then every solution $(x(t),y(t))$ of the system (4.4) satisfies

$$\lim_{t\to\infty} x(t) = 0; \qquad \lim_{t\to\infty} y(t) = 0.$$

Example 4.3. Consider the second-order nonlinear non-autonomous stochastic differential equation

$$\begin{aligned} x'' + (3+2t+|\cos(xx')|)x' + (x+\sin x) + 0.11x\omega'(t) \\ = (1+t+xx')^{-1}. \end{aligned} \tag{4.5}$$

Equation (4.5) is equivalent to system

$$\begin{aligned} & x' = y, \\ & y' = (1+t+xy)^{-1} - (x+\sin x) - (3+2t+|\cos(xy)|)y \\ & -0.11x\omega'(t). \end{aligned} \tag{4.6}$$

Now from system (3.36) and (4.6), we have the following relations:

(i) The function $g(t,x,x') = 3+2t+|\cos(xy)|$. And since $|\cos(xy)| \geq 0$ for all x and y, it follows from hypothesis (i) of Theorem 3.6.

$$g(t,x,y) = 3+2t+|\cos(xy)| \geq a = 3,\ t \geq 0,\ x \text{ and } y.$$

(ii) The function $f(x) = x+\sin x,\ x \neq 0$. From hypothesis (ii) of Theorem 3.6 $\dfrac{f(x)}{x} = 1+\dfrac{\sin x}{x},\quad x \neq 0$. It follows that $\dfrac{f(x)}{x} = 1+\dfrac{\sin(x)}{x} \geq b = 0.8$. Thus,

$$0.8 = b \leq \frac{f(x)}{x} = 1+\frac{\sin(x)}{x} \leq \beta = 2 \quad x \neq 0.$$

And of course hypothesis (iii) of Theorem 3.6 is satisfied

(iii) The function $p(t,x,y) = [1+t+xy]^{-1}$. And since $1+t+|xy| \geq 1,\ t \geq 0,\ x$ and y, then $|p(t,x,y)| \leq B_0 = 1$ for all $t \geq 0, x$ and y.

Items (i) to (iii) fulfill all hypotheses of Theorem 3.6, hence by Theorem 3.6 solution $(x(t),y(t))$ of the stochastic differential system (4.6) is uniformly stochastically bounded. In addition to Theorem 3.6

(iv) The function $\int_0^\infty \frac{1}{1+t+|xy|}dt < \beta < \infty$ for all $t \geq 0$, x and y.

Then every solution $(x(t), y(t))$ of the system (4.6) satisfies

$$\lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} y(t) = 0.$$

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