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# A THREE-STEP SIMPSON'S TYPE EXPONENTIALLY-FITTED BACKWARD DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, a class of exponential fitting backward difference method (EFBDM) is derived for numerically solving general first order ordinary differential equations. This class of method is from the linear multistep method (LMM) derived via the technique of collocation. The power series polynomial used as basis function is fitted with an exponential function term. This class of EFBDM is derived for the step number k=3. The method satisfies the basic features of numerical scheme which includes consistency and zero-stability. The convergence of the method is also established. This class of 3-step method is compared to already existing methods in literature to establish its efficiency in terms of global errors and they compare favourably with the methods cited.

## 1. INTRODUCTION

In this work, we consider the first order Initial Value Problem (IVP) of the form

$$\left.\begin{aligned} &y' = f(x,y), \quad a < x < b, \\ &y(a) = \alpha_0 \end{aligned}\right\} \tag{1.1}$$

where $\alpha_0$ is an arbitrary finite real constant, $x \in (a,b)$, $f \in C[a,b]$, is a continuous function defined on the interval $(a,b)$.

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Several physical processes in science and engineering can be modeled in the form (1.1) which could be non-stiff, stiff or singular in nature. Specifically, we consider non-stiff and singular IVPs that typically originate from models in Mathematical Biology triggered by a random walk of organisms, such as bacterial in the presence of a chemical resulting in interesting spatial patterns (see Tyson *et al.* [18]). We note that the solutions of the non-stiff or singular ODEs may become unbounded in finite time and such a phenomenon is often called blow-up, while the finite time is called the blow-up time (see [19]-[21]).

The idea behind exponential fitting (EF) is to derive numerical methods that are better suited for oscillatory and stiff problems. Classical method with the basis $1,x,x^2,\dots,x^k$ perform best when the solution is a polynomial, as a k-step Adams-Bashforth method can even find a polynomial solution of degree $k$ without errors and up to machine accuracy. The idea of using exponentially fitted formulae for numerically solving differential equations came from Liniger and Willoughby [22], where integration formulae containing free parameters were derived and these parameters were chosen so that the function $exp(w)$ with $w$ real, satisfied the integration formulae. This was tested on linear and used on multistep method for $k=1$. From this, many others have developed exponentially-fitted method capable of handling many types of problems especially the nonlinear and oscillatory ones.

To obtain an exponentially fitted variant method, a few of the highest-order monomials are replaced by exponentials. The most general fitting space is of the form $\{1,x,x^2,\dots,x^k,e^{w_0x},e^{w_1x},\dots,e^{w_px}\}$ which solely depend on the parameters $w_0,\dots,w_p$ multiplied by the step-size $h$, see Ixaru and Paternoster [7], Ixaru and Vanden [8]. Particularly, the parameters $w_0,\dots,w_p$ can all be given different values. It can however be interesting to specify a relation between the different parameters. Regardless of the form of the fitting space, it is usually imposed that the parameter value(s) are either real or imaginary, (see [11]-[16]). Here, the exponential fitting space of the form $\{1,x,x^2,x^3,e^{wx}\}$ is used for the derivation of the exponentially fitted method.

## 2. Existence and Uniqueness of First Order ODEs

The following theorem guarantees the existence of at least one solution of (1.1).

**2.1 Existence:** Suppose that $f(x,y(x))$ is a continuous function defined in some region

$$R=\{(x,y(x)):x_0-\delta<x<x_0+\delta,\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

containing the point $(x_0, y_0)$. Then there exists a number $\delta_1 < \delta$ such that a solution y = $f(x)$ of (1.1) is defined for $x_0 - \delta_1 < x < x_0 + \delta_1$,

**2.2 Uniqueness:** Suppose that both $f(x, y(x))$ and $\frac{\partial f(x,y(x))}{\partial y}$ are continuous functions defined on the region

$$R = \{(x, y(x)) : x_0 - \delta < x < x_0 + \delta,\ \ y_0 - \varepsilon < y < y_0 + \varepsilon\}$$

Then there exists a number $\delta_2 < \delta_1$ such that the solution $y = f(x)$ of (1.1) whose existence was guaranteed by the existence theorem, has a unique solution for $x_0 - \delta_2 < x < x_0 + \delta_2$, see Brugnano and Trigiante, [23]-[24].

## 3. Development of the Method

For a 3-step method, we consider the interval $[x_n, x_{n+3}]$, where $x_{n+i} = x_n + ih$, $h = x_j - x_{j-1}$, for solving the problem in (1.1) on the interval $[a,\ b]$. We consider the approximation of its solution $y(x)$ by a polynomial $u(x)$ given by

$$y(x) \approx u(x) = \sum_{i=0}^{3} a_i x^i + a_4 e^{wx} \tag{3.1}$$

whose derivative is given as

$$y'(x) \approx u'(x) = \sum_{i=1}^{3} i\ a_i x^{i-1} + w a_4 e^{wx} \tag{3.2}$$

with the $a_i \in \mathbb{R}$ real unknown parameters to be determined, and the parameter $w$ will be real constant. Imposing appropriate interpolation conditions to $u(x)$ at $x_n$ and collocation condition $u'(x)$ at the points $x_n, \ldots, x_{n+3}$, leads to the system of 4 equations:
$u(x_n) = y_n$, $u'(x_n + ih) = u'_{n+i} = f_{n+i}$; $i = 0(1)(3)$ for the determination of $a_i$'s, $i = 0(1)(3)$. Solving the system and obtaining the constants $a_0, a_1, a_2, a_3$, which are then substituted into (3.1). The formula of the form

$$u(x) = \alpha_0 y_n + \sum_{i=0}^{3} \beta(w;h) f_{n+i} \tag{3.3}$$

is derived, where $\alpha_0 = 1$, $\beta$ is a function depending on $w$ and $h$. Evaluating (3.3) at the point $x_{n+i}$, $i = 0(1)3$, the following 3-step EFBDM are obtained:

$$
\begin{aligned}
y_{n+1} = {} & y_n + \eta_1 \big((12 - 16e^{2hw}hw + 5e^{3hw}hw + e^{hw}(-12 + 23hw))f_n + (-36 + 36e^{hw} - 23hw - 21e^{2hw}hw + 8e^{3hw}hw)f_{n+1} \\
& -(e^{3hw}hw + e^{hw}(36 - 21hw) - 4(9 + 4hw))f_{n+2} + (-12 - 5hw + e^{2hw}hw + e^{hw}(12 - 8hw))f_{n+3}\big) \\
y_{n+2} = {} & y_n + \eta_2 \big((3 + 7e^{hw}hw + e^{3hw}hw - e^{2hw}(3 + 2hw))f_n + (-9 - 7hw + 4e^{3hw}hw + e^{2hw}(9 - 15hw))f_{n+1} \\
& +(9 - 9e^{2hw} + 2hw + 15e^{hw}hw + e^{3hw}hw)f_{n+2} + (-3 + 3e^{2hw} - hw - 4e^{hw}hw - e^{2hw}hw)f_{n+3}\big) \\
y_{n+3} = {} & y_n + \eta_1 \big(9(-4 + 4e^{3hw} - 3hw)f_n - (9e^{2hw}hw)f_{n+1} + (9(4 + 9e^{hw}hw + e^{3hw}(-4 + 3hw)))f_{n+2} \\
& +3(-4 + 4e^{3hw} - 3hw - 9e^{2hw}hw)f_{n+3}\big)
\end{aligned}
\tag{3.4}
$$

$$\eta_1 = \frac{1}{12w(e^{hw}+1)^3},\ \eta_2 = \frac{1}{3w\left(e^{hw}+1\right)^3 w},$$

## 3.1. **Analysis of the Method.**

Local truncation error (LTE) and order

The linear difference operators associated with the formulas in (3.4), is given as

$$\mathscr{L}_k[y(x);\, h] \equiv y(x+nh) - y(x) - \left[\sum_{i=1}^{3} \beta_i(w;h)y'(x+ih)\right] \tag{3.5}$$

The local truncation error of each of the formulae in (3.4) is the amount by which the exact solution of the ODEs fails to satisfy the corresponding difference operator. Thus, if we consider the exact solution $y(x)$ in (3.5), after expanding in Taylor series around $x$ we get that each of the local truncation errors of the form

$$\mathscr{L}[y(x);\, h] = C_{p+1}h^{p+1}y^{(p+1)}(x) + O(h^{p+2}) \tag{3.6}$$

where the constants $C_i$ for $i = 0(1)p$ varnishes. The $C_{p+1}$ is called the principal error constant and $p$ is called the order of the formula. In this case, for all the formulae in (3.4), the form of (3.6) are given as

$$C_{p+1} = \begin{pmatrix} -\frac{95}{288}\left(wy^{(5)}(x) - y^{(6)}(x)\right)h^6 + O(h)^7 \\ -\frac{14}{45}\left(wy^{(5)}[x] - y^{(6)}(x)\right)h^6 + O(h)^7 \\ -\frac{51}{160}\left(wy^{(5)}[x] - y^{(6)}(x)\right)h^6 + O(h)^7 \end{pmatrix}$$

Conclusively, the order $p = 5$.

## 3.2. **Zero-stability.**

**Definition 3.2.1:** The first characteristic polynomial of k-step linear multistep method is the degree-k polynomial

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{3.7}$$

**Definition 3.2.2:** A polynomial is said to satisfy the root condition if all its roots lie within or on the unit circle, with those on the boundary being simple. In other words, all roots satisfy $|z| \leq 1$ and any that satisfy $|z| = 1$ are simple. A polynomial satisfies the strict root condition if all its roots lie inside the unit circle; that is, $|r| < 1$, Amodio and Mazzia, [1]. [1]

**Definition 3.2.3:** A Linear Multistep Method is said to be zero-stable if its first characteristic polynomial $\rho(z)$ satisfies the root condition.

The methods in (3.4) if put the block form can be written as

$$A_0 Y_{r+n} = A_1 Y_r + hB(h;w)F_{r+n} \tag{3.8}$$

The zero-stability is concerned with the stability of the difference system (3.8) as $h \to 0$. Thus, as $h \to 0$, (3.8) becomes

$$A_0 Y_{r+n} = A_1 Y_r \tag{3.9}$$

where

$$Y_{r+n} = (y_{n+1}, y_{n+2}, y_{n+3})^T \,, \quad Y_r = (y_n, y_{n-1}, y_{n-2})^T$$

$A_0$ is a 3 by 3 identity matrix written as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Hence we sought for the characteristic polynomial

$$\rho(z) = |zA_0 - A_1| = 0 \tag{3.10}$$

such that the roots $\rho(z) = z^2(z-1) = 0$, $z = 1, 0, 0$. Consequently, the method is zero-stable, since the roots of the characteristic polynomial are all zero except one, whose modulus is one (see Dahlquist [3], Lambert [10]). For convergence, we state the following theorem.

## 3.3. **Convergence.**

**Theorem 3.3.1** Henrici [5]. A linear multistep method is said to be convergent if it is consistent (with order $p \geq 1$) and it is zero-stable.

By the above analysis, the method has order $p > 1$, and is zero-stable. Thus, by the above theorem, the method is convergent.

[1] We say that $\lambda$ is a simple root of $\rho(z)$ if $\lambda - z$ is a factor of $\rho(z)$.

### 3.4. Implementation of EFBDM.

The block method is implemented as follows: Using method (3.8), $n = 0$, solve for the values of $y_1$ with the aid of Newton's Method on the sub-interval $[x_0, x_1]$, as $y_0$ is known from the IVP (1.1). Next, for $n = 1$, the values of $y_2$ is obtained over the sub-interval $[x_1, x_2]$, as $y_1$ is known from the previous sub-interval. The process is continued for $n = 2, \ldots, N-1$ to obtain the numerical solution of (1.1) on the sub-intervals $[x_2, x_3]$, $[x_3, x_4], \ldots, [x_{N-1}, x_N]$. It should also be noted that the frequency $w$ is determined by the exponential term in the exact solution. Other values can be used in a case where the exact solution is unknown. The details of the implementation is given in Algorithm below.

**Algorithm 1** Block Algorithm for EFBDM

1 **begin procedure**ENTER Partitions $(a, b, N, h, \text{variables})$
2 For $x_n = x_{n-1} + h$, $n = 1, \ldots, N$, $h = \frac{b-a}{N}$
3 Generate block system
4 Solve [sysytem, variables]
5 Obtain $y_n$
6 **end procedure**

## 4. Numerical Examples

In this section, we give numerical examples to illustrate the accuracy of the method. Let $y(x_n)$ be the exact solution and $y_n$ the approximate solution on the partition $\pi_N$, we find the absolute errors of the approximate solution as $|y(x_n) - y_n|$

**Problem 1.** We consider the linear IVP, [2].

$$\begin{cases} y'^2, & 0 \le x \le 1 \\ y(0) = 1 \end{cases} \tag{4.1}$$

The analytic solution is given by $y(x) = -2 - 2x - x^2 + 3e^x$;. With $N = 10$, $w = -1$.

From the above table, our method become superior over the method in [2].

**Problem 2.** We consider the singular IVP discussed in [9].

$$\begin{cases} y'^2 = 1, & 0 \le x \le 1 \\ y(0) = 1 \end{cases} \tag{4.2}$$

The analytic solution is given by $y(x) = \tan\left(x + \frac{\pi}{4}\right)$. With $N = 20$.

TABLE 1. Comparison of absolute errors obtained in different methods for Example 1.

| $x$ | $y_{app}$ | $y_{ex}$ | EFBDM | Method in [2] |
|---|---|---|---|---|
| 0.1 | 1.105512754 | 1.105512754 | 1.77E-15 | 2.45E-5 |
| 0.2 | 1.224208274 | 1.224208274 | 1.73E-14 | 2.71E-5 |
| 0.3 | 1.359576422 | 1.359576422 | 7.11E-15 | 2.99E-5 |
| 0.4 | 1.515474092 | 1.515474092 | 2.66E-14 | 3.31E-5 |
| 0.5 | 1.696163812 | 1.696163812 | 4.79E-14 | 3.65E-5 |
| 0.6 | 1.906356401 | 1.906356401 | 1.51E-14 | 4.04E-5 |
| 0.7 | 2.151258122 | 2.151258122 | 7.11E-15 | 4.46E-5 |
| 0.8 | 2.436622785 | 2.436622785 | 3.91E-14 | 4.93E-5 |
| 0.9 | 2.768809333 | 2.768809333 | 6.75E-14 | 5.45E-5 |
| 1.0 | 3.154845485 | 3.154845485 | 8.17E-14 | 6.03E-5 |

TABLE 2. Comparison of absolute errors for Example 2.

| $x$ | EFBDM | PCM | SSM | RKM |
|---|---|---|---|---|
| 0.0 | 0 | 0.00 | 0.00 | 0.00 |
| 0.1 | 6.71(-10) | 1.06(-6) | 1.03(-6) | 2.15(-8) |
| 0.2 | 7.88(-9) | 2.71(-6) | 2.45(-6) | 2.80(-8) |
| 0.3 | 1.63(-8) | 5.37(-6) | 4.68(-6) | 5.21(-7) |
| 0.4 | 2.09(-8) | 1.03(-5) | 8.71(-6) | 3.63(-6) |
| 0.5 | 2.65(-8) | 2.13(-5) | 1.75(-5) | 2.60(-5) |
| 0.6 | 2.77(-8) | 5.54(-5) | 4.41(-5) | 2.91(-4) |
| 0.7 | 1.56(-8) | 2.80(-4) | 2.14(-4) | 1.34(-2) |
| 0.8 | 3.05(-8) | 1.02(-2) | 7.37(-3) | 1.39(-3) |
| 0.9 | 1.87(-7) | 1.58(-4) | 1.18(-4) | $\infty$ |
| 1.0 | 7.46(-7) | 4.43(-5) | 3.22(-5) | $\infty$ |

NSDM is nonlinear one-step second derivative method with a variable step-size implementation based on continued fractions for the numerical solution of singular initial value problems (IVPs) in [9]. The method was implemented in Predictor-Corrector Mode (PCM) and Self-Starting Mode (SSM). Where RKM is the Runge-Kutta Method for the same problem. It can be clearly seen that our method compares favourably. This shows the superiority of our method.

**Problem 3.** We consider a linear system of IVP, [17].

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -2y_1(x)+y_2(x) \\ 998y_1(x)-999y_2(x) \end{pmatrix} + \begin{pmatrix} 2\sin(x) \\ 999(\cos(x)-\sin(x)) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.3}$$

TABLE 3. EFBDM absolute errors for Example 3.

| $x$ | $y_1$ Approx | $y_1$ Exact | $y_2$ Approx | $y_2$ Exact | Error 1 | Error 2 |
|---|---|---|---|---|---|---|
| 5.0 | -0.9454483808 | -0.9454483808 | 0.2971380792 | 0.2971380792 | 2.15E-10 | 2.17E-10 |
| 50.0 | -0.26237485419 | -0.2623755419 | 0.9649660279 | 0.9649660279 | 4.92E-10 | 4.97E-10 |

TABLE 4. Errors for Example 3 in [17]

| $x$ | Error 1 of Method 1 | Error 2 of Method 1 | Error 1 of Method 2 | Error 2 of Method 2 |
|---|---|---|---|---|
| 5.0 | 1.27E-3 | 1.35E-6 | 1.19E-3 | 1.26E-6 |
| 50.0 | 3.70E-5 | 1.10E-7 | 3.27E-5 | 1.02E-7 |

TABLE 5. Maximum errors obtained for Example 4.

| Methods | $Max\{y_1, y_2\}$ |
|---|---|
| EFBDM | 9.21E-20 |
| BLOCK10SIMP | 1.00E-14 |
| BLOCK10 | 1.00E-15 |
| BLOCK2SIMP | 1.00E-6 |
| BLOCK2 | 1.00E-6 |

The exact solution is

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 2e^{-x} + \sin(x) \\ 2e^{-x} - \cos(x) \end{pmatrix} \tag{4.4}$$

Here, $w = 1$.

We remark that EFBDM is very competitive with both methods discussed in [17]. Hence, this shows the superiority of our methods in terms of the errors obtained.

**Problem 4.** Consider the following system of first order IVP discussed in [6].

$$\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2, \ \ y_1(0) = 1 \\ y_2'(x) &= y_1 - y_2(1 + y_2), \ \ y_2(0) = 1 \end{aligned} \tag{4.5}$$

The exact solution are; $y_1(x) = e^{-2x}$, $y_2(x) = e^{-x}$, $w = 2$.

The above Table 5 shows the maximum error of $y_1$ and $y_2$, that is, $Max\{y_1, y_2\}$.

Here, in [6], BLOCK2 is the 2-step block method, BLOCK2SIMP is the simplest 2-step block method, BLOCK10 is the 10-step block method, BLOCK10SIMP is the simplest 10-step block method. The proposed method performed better than the methods compared with in [6].

TABLE 6. Absolute errors for Example 5.

| $h$ | Method 1 in [25] | Method 2 in [25] | Method 3 in [25] | EFBDM |
|---|---|---|---|---|
| $10^{-10}$ | $5.20713 \times 10^{-10}$ | $1.02940 \times 10^{-12}$ | $5.06390 \times 10^{-12}$ | $1.24575 \times 10^{-16}$ |
| $10^{-11}$ | $1.67741 \times 10^{-10}$ | $2.72390 \times 10^{-12}$ | $4.40500 \times 10^{-13}$ | $3.42518 \times 10^{-17}$ |
| $10^{-12}$ | $5.33553 \times 10^{-11}$ | $1.16940 \times 10^{-12}$ | $2.01000 \times 10^{-14}$ | $4.15785 \times 10^{-17}$ |
| $10^{-13}$ | $1.68998 \times 10^{-11}$ | $3.96500 \times 10^{-13}$ | $1.40000 \times 10^{-15}$ | $1.27154 \times 10^{-18}$ |

**Problem 5.** Consider a first order nonlinear singular IVP solved in [6].

$$y' = -\frac{y^2}{\sqrt{x}}, \;\; y(0) = 1, \;\; 0 < x \leq 1 \tag{4.6}$$

The exact solution is given by $y(x) = \frac{1}{1+2\sqrt{x}}$. We thus solve the initial value problem (4.6) by adopting the same value for the step-size $h = 1/12000000$ as in [6].

Table 6 shows, that for very small $h$, EFBDM produced excellent results than methods proposed in [25]. It was keenly observed that, even as $x$ is very close to singular 0, the results are not affected. Hence, the proposed EFBDM solved the problem and the results obtained are more superior in terms of the error obtained when compared to the errors obtained in [25].

To show the accuracy of the EFBDM, we solve Susceptible, Exposed, Infective and Recovery (SEIR) tuberculosis disease model in [26] with EFBDM and compare graphically with Runge-Kutta method of order 4.

**Problem 6.** Consider the SEIR tuberculosis disease model discussed in [26].

$$\begin{aligned} \frac{ds}{dt} &= \mu - \mu s - \beta s i \\ \frac{de}{dt} &= \beta s i - (\mu + \varepsilon) e \\ \frac{di}{dt} &= \varepsilon e - (\mu + \gamma) i \\ \frac{dr}{dt} &= \gamma i - \mu r \end{aligned} \tag{4.7}$$

where $s = \frac{S}{N}$, $e = \frac{E}{N}$, $i = \frac{I}{N}$, and $r = \frac{R}{N}$, represents the fractions of the susceptible $S$, exposed $E$, infective $I$ and recovery $R$ classes in the population respectively, with initial conditions $s(0) = s_0$, $e(0) = e_0$, $i(0) = i_0$, $r(0) = r_0$. The following are the parameters used for numerical simulation: $s(0) = 5000$, $e(0) = 1000$, $i(0) = 150$, $r(0) = 50$, $\beta = 0.0468$, $\varepsilon = 0.1196$, $\gamma = 0.1472$, $\mu = 0.0006$. The figures below show the comparison of EFBDM and the fourth order Runge-Kutta method.

The proposed method was used with appropriate step-size $h$ depending on the method being compared with and the frequency $w = 1$. The

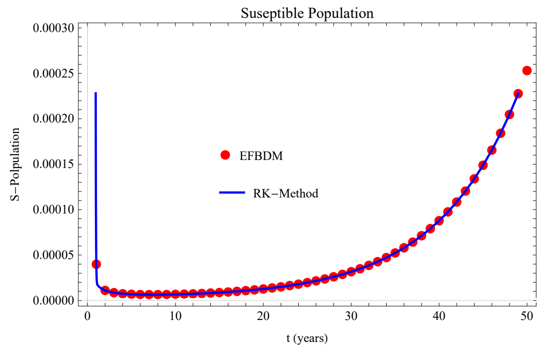


FIGURE 1. Susceptible Population

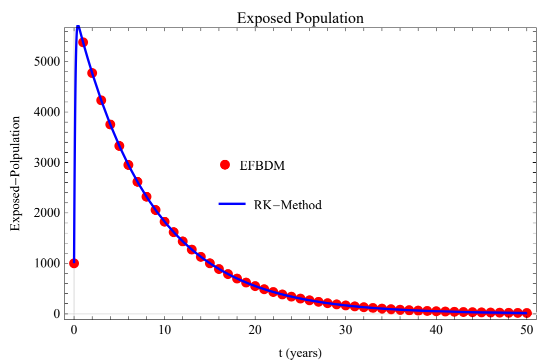


FIGURE 2. Exposed Population

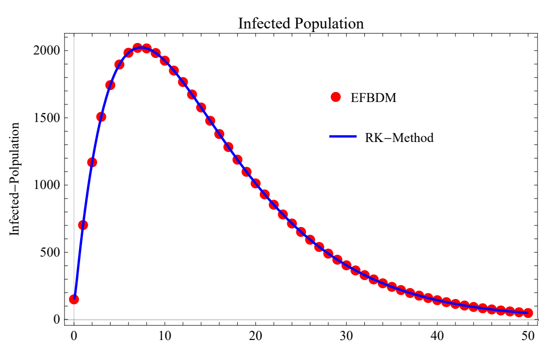


FIGURE 3. Infective Population

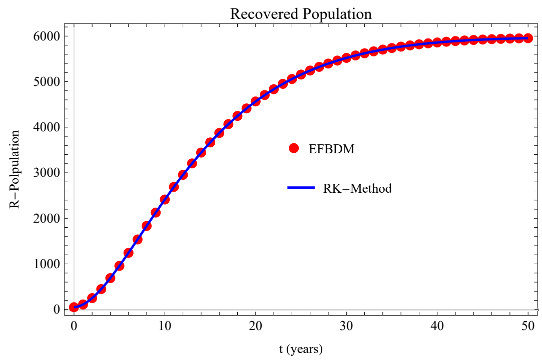


FIGURE 4. Recoverable Population

performance of the EFBDM has been demonstrated to outperform other methods compared with in the cited literature evidently from Problems 1 through 5.

## 5. CONCLUSION

An Exponentially-fitted Backward Difference Formula (EFBDM) based on continuous linear multistep method is proposed and applied to solve first order linear and non linear IVPs in ordinary differential equations. It can be seen that the method is very easy to derive and less ambiguous. It can be applied to solve a wide range of first order ODEs as seen in the numerical examples. The method shows a very high accuracy when compared to the exact solution and existing methods in the literature.

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## REFERENCES

[1] P. Amodio and F. Mazzia, *A boundary value approach to the numerical solution of initial value problems by multistep methods*, J. of Difference Equations and Applications, **1**, 353-367, 1995.

[2] S. O. Ayinde and E. A. Ibijola, *A New Numerical Method for Solving First Order Differential Equations*, American Journal of Applied Mathematics and Statistics, **3** (4) 156-160, 2015.

[3] G. Dahlquist, *Convergence and stability in the numerical integration of ordinary differential equations*, Math. Scand. **4**, 33-53, 1956.

[4] H. De Meyer, J. Vanthournout and G. Vanden Berghe, *On a new type of mixed interpolation*, Journal of Computational and Applied Mathematics, **30**,(1), 55-69, 1990.

[5] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1962.

[6] H. Ramos and P. Popescu, *How many k-step linear block methods exist and which of them is the most efficient and simplest one*, Applied Mathematics and Computation, **316**, 296-309, 2018.

[7] L G. Ixaru and B. A. Paternoster, *Conditionally P-stable fourth-order exponential-fitting method for y' = f (x, y)*, Journal of Computational and Applied Mathematics, **106** (1) 87-98, 1999.

[8] L G. Ixaru and B. G. Vanden, *Exponential Fitting*, **568** of Mathematics and Its Applications. Kluwer Academic Publishers, Belgium, 2004.

[9] S. N. Jator and N. Coleman, *A nonlinear second derivative method with a variable step-size based on continued fractions for singular initial value problems*, Cogent Mathematics,**4** 1335498, 2017.

[10] J. D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley, New York, 1973.

[11] T. E. Simos, *Some New Four-Step Exponential-Fitting Methods for the Numerical Solution of the Radical Schrödinger Equation*, IMA Journal of Numerical Analysis **11** (3) 347-356, 1991.

[12] T. E. Simos, *Exponential fitted methods for the numerical integration of the Schrödinger equation*, Computer physics communications **71** (1) 32-38, 1992.

[13] T. E. Simos, *Error analysis of exponential-fitted methods for the numerical solution of the one-dimensional Schrödinger equation*. Physics Letters, **177**, (4) 345-350, 1993

[14] G. Vanden Berghe, H. De Meyer and J. Vanthournout, *A modified Numerov integration method for second order periodic initial-value problems*, International Journal of Computer Mathematics **32**, (3) 233-242, 1990.

[15] G. Vanden Berghe and M. Van Daele, *Exponentially-fitted Numerov methods*, Journal of Computational and Applied Mathematics, **200** (1) 140-153, 2007.

[16] J. Vanthournout, G. Vanden Berghe and H. De Meyer, *Families of backward differentiation methods based on a new type of mixed interpolation*, Computers and Mathematics with Applications **20**, (11) 19-30, 1990.

[17] D. G. Yakubu, M. Aminu, P. Tumba and M. Abdulhameed, *An Efficient Family Of Second Derivative Runge-Kutta Collocation*

*Methods For Oscillatory Systems*, Journal of the Nigerian Mathematical Society, **37** (2) 111-138, 2018.

[18] R. Tyson, L. G. Stern and R. J. LeVeque, *Fractional step methods applied to a chemotaxis model*, Journal of Mathematical Biology, **41**, 455-475, 2000.

[19] C. Cho, *On the computation of the numerical blow-up time*. Japan Journal of Industrial and Applied Mathematics, **30**, 331-349, 2013.

[20] P. Groisman, *Totally discrete explicit and semi-implicit Euler methods for a blow-up problem in several space dimensions*, Computing, **76**, 325-352, 2006.

[21] A. Takayasu, K. Matsue, T. Sasaki, K. Tanaka, M. Mizuguchi and S. Oishi, *Numerical validation of blow-up solutions of ordinary differential equations*. Journal of Computational and Applied Mathematics, **314**, 10-29, 2017.

[22] W. Liniger and R. A. Willoughby, *Efficient Integration Methods for Stiff Systems of Ordinary Differential Equations*, SIAM Journal on Numerical Analysis, **7** (1) 47-66, 1970.

[23] L. Brugnano and D. Trigiante, *Convergence and stability of boundary value methods for ordinary differential equations*, Journal of Computational and Applied Mathematics, **66** 97-109, 1996.

[24] L. Brugnano and D. Trigiante, *Solving differential problems by multistep initial and boundary value methods*, Gordon and Breach Science Publishers, Amsterdam, 1998.

[25] M. H. Rahaman, M. K. Hasan, M. Ayub Ali and M. Shamsul Alam, *Implicit Methods for Numerical Solution of Singular Initial Value Problems*, Applied Mathematics and Nonlinear Sciences **6** (1) 1–8, (2021)

[26] K. Das, B. S. N. Murthy, S. A. Samad and M. H. A. Biswas, Mathematical transmission analysis of SEIR tuberculosis disease model, Sensors International 2 (2021) 100120, www.keaipublishing.com/en/journals/sensors-international

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