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QUASI-CENTRAL PRODUCT OF GROUPS AS A GENERALIZATION OF CERTAIN PRODUCT OF GROUPS

A. IGUDA[1], M. M. ZUBAIRU[2], A. I. KIRI[3] AND M. J. BASHIR[4]

ABSTRACT. In this paper we relaxed the condition $[H,K] = \{e\}$ in the definition of central product and come up with a new product called *quasi-central product*. We claim and prove that every central product is quasi-central product but not vice versa. We defined both external and internal quasi-central products and further show that the external and internal quasi-central products are isomorphic.

1. INTRODUCTION

Let G be a group and H and K be subgroups of G. The subgroups H and K are said to be *essentially disjoint* in G if $H \cap K = \{e\}$. H is said to be a *complement* of K in G if H and K are essentially disjoint and $G = HK$. A group G is said to be the *internal direct product* (resp., *internal semidirect product*) of H and K if both H and K are normal (resp., H is normal and K not necessary normal) and H and K are complements in G. For two groups G_1 and G_2, the set $G_1 \times G_2 = \{(g_1,\ g_2) : g_1 \in G_1,\ g_2 \in G_2\}$ together with the binary operation defined by $(g_1,\ g_2)(g_1',\ g_2') = (g_1g_1',\ g_2g_2')$ is known to be the *external direct product* of G_1 and G_2. Similarly if there exists a homomorphism ϕ from G_2 to the set of automorphisms of G_1 (i.e., $\phi : G_2 \to \mathrm{Aut}(G_1)$), then the set $G_1 \rtimes_\phi G_2$ together with the binary operation defined by $(g_1,\ g_2)(g_1',\ g_2') = (g_1\phi_{g_2}(g_1'),\ g_2g_2')$ is known to be the *external semidirect product* of G_1 and G_2. It is well known that in both direct and semidirect products, the external and internal products are isomorphic. Moreover, if $G = H \rtimes_\phi K$ (resp., if $G = H \times K$), then G contains a normal subgroup isomorphic to H and a subgroup isomorphic to K (resp., G contains normal subgroups isomorphic to H and K),

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see [6, 10] for more details. A group G is an *internal central product* of its subgroups H and K if $G = HK$, $H \cap K$ is the center of G and $[H,K] = \{e\}$, where $[H,K] = \{h^{-1}k^{-1}hk : h \in H, k \in K\}$ (i.e., the set of commutators of H and K).

Let G_1 and G_2 be groups and let A and B be the centers of G_1 and G_2, respectively, such that there exists an isomorphism φ from A to B. The *external central product* of G_1 and G_2 is given by $G_1 o_\varphi G_2 = \{(g_1,\ g_2)N : g_1 \in G_1,\ g_2 \in G_2\}$, where $N = \{(a,\ \varphi(a^{-1})) :\ a \in A\}$, $\varphi(a^{-1}) \in B$ with operation defined by $(g_1,\ g_2)N(g_1',\ g_2')N = (g_1g_1',\ g_2g_2')N$, for $(g_1,\ g_2)N,\ (g_1',\ g_2')N \in G_1 o_\varphi G_2$. As in the direct and semidirect products of groups, the internal and external central products are also isomorphic, for more detail results we refer the reader to [7]. Further, if $G = HoK$, then G is generated by normal subgroups isomorphic to H and K, and all elements of H commute with all elements of K.

Throughout this paper we shall write G and $|G|$ to denote a *group* and *order of a group* G, respectively. Further, e, $H \leq G$ and $Z(G)$ denote, the *identity*, *H is a subgroup of G* and *center of the group G*, respectively. Moreover, for two subgroups H and K of a group G, HK, $H \rtimes K$ and HoK denote the *internal direct product* of H and K, the *internal semi-direct product* of H and K and internal central product of H and K, respectively. Similarly, $G_1 \times G_2$, $G_1 \rtimes_\varphi G_2$ and $G_1 o_\varphi G_2$ denote the *external direct product* of groups G_1 and G_2, the *external semi-direct product* of groups G_1 and G_2 and *external central product* of groups G_1 and G_2, respectively. For basic concepts in product of groups and other undefined terms, we refer the reader to [4, 6, 9, 10].

In 1852, Betti [8] defined what is now known as direct product of groups, he used groups of substitution in his work. The concept of the groups product in Betti's paper was not very clear at that time. After the modern definition of group, many results followed about direct product of groups for example see [9, 10]. In 1937, Schur and Zassenhaus showed that for a group G of order nm where n and m are co-prime, if G has a normal subgroup H of order n then H has complement say K in G [11].

Furthermore, Neumann and Neumann [2], in 1950 defined another product called central product as earlier defined in the introductory paragraph. In this product, G is the central product of H and K only if H and K generate G (i.e., $G = HK$), $H \cap K = Z(G)$ and all the elements of H commute with all elements of K (i.e., $[H,K] = \{e\}$). We have noticed that the condition $[H,K] = \{e\}$ in the definition of central product is too strong. We ask a question that what if $\{e\} \subsetneq [H,K]$ and $H \cap K = Z(G)$, can we still have $G = HK$? The answer to this question is that despite

$\{e\} \subsetneq [H,K]$ and $H \cap K = Z(G)$, we can still have $G = HK$. This is what we are going to discuss in this paper.

To verify our claim we consider the dicyclic group of order 12, (i.e., $Dic_{12} = \langle h,k : h^6 = k^4 = e, h^3 = k^2, kh = h^{-1}k\rangle$). It is easy to verify that $K = \langle k : k^4 = e\rangle$ and $H = \langle h : h^6 = e\rangle$ are subgroups of Dic_{12}. Moreover, $H \cap K = \{e,h^3\} = Z(Dic_{12})$. Notice that $[H,K] = \{e,h^2,h^4\} \neq \{e\}$, thus by definition G is not the central direct product of H and K. We further notice also that the subgroups H and K are not essentially disjoint in Dic_{12} and as such Dic_{12} is not a semidirect product of H and K. However, one can easily verify that $G = HK$. This counter example gives us a group with two subgroups with their intersection the center of the group and $[H,K]$ not singleton set containing identity only, and yet their product is the whole group. This example has proved our claim of superfluousness of the condition $[H,K] = \{e\}$.

The condition $[H,K]$ is not necessarily $\{e\}$ is achievable when we can find an element in H and another element in K which do not commute. This means the set $[H,K]$ contains some element apart from identity, (i.e., $\{e\} \subset [H,K]$). For example, in the dicyclic group of order 12, the set $[H,K] = \{e,h^2,h^4\}$. This means the set of commutators of H and K properly contained $\{e\}$ (i.e., $\{e\} \subset [H,K]$). Therefore Dic_{12} cannot be a central product of H and K, also since H and K are not essentially disjoint in Dic_{12} then also Dic_{12} cannot be the semidirect product of H and K.

Thus in this paper we relax the condition $[H,K] = \{e\}$ of the definition of central product and defined a new product called *quasi-central product*. We claimed that every central product is quasi-central products but not vice versa. We defined both external and internal quasi-central products and we further show that, the external and internal quasi-central products are isomorphic.

2. INTERNAL QUASI-CENTRAL PRODUCT OF GROUPS

In this section we introduce the *quasi-central product* of groups and we show that this product is a group. Now by virtue of the counter example given in the previous section, we give the following non-vacuum definition.

Definition 2.1. Let G be a group, $H,K \leq G$. Then G is said to be the *internal quasi-central product* of H and K if

(i) $G = HK$;
(ii) $H \cap K \leq Z(G)$;
(iii) $H \lhd G$ or $K \lhd G$.

We denote internal quasi-central product of H and K by $H\tilde{o}K$. As an immediate consequence of the above definition, we have the following result.

Theorem 2.2. *Internal quasi-central product of two subgroups is a group.*

The following corollary follows from the above theorem.

Corollary 2.3. *Every internal central product is internal quasi-central product, but the converse is not necessarily true.*

Proof. Let G be the internal central product of H and K, where $H, K \leq G$. Thus, $G = HK$, $H \cap K = Z(G)$ and $[H,K] = \{e\}$. To show G is quasi central product of H and K, we only need to show either $H \lhd G$ or $K \lhd G$. To show $H \lhd G$, let $g \in G, h \in H$ then $g = h'k'$, for some $h' \in H$ and $k' \in K$. Notice that $ghg^{-1} = h'k'h(h'k')^{-1} = h'k'hk'^{-1}h'^{-1}$. Now since $[H,K] = \{e\}$, it means that $k'hk'^{-1}h'^{-1} = e$, i.e., $k'h = hk'$. Thus, $h'k'hk'^{-1}h'^{-1} \in H$. i.e., $ghg'^{-1} \in H$. Therefore $H \lhd G$ and as such G is quasi central product of H and K, as required.

The counter example in section 1 disprove the converse. □

Corollary 2.4. *Every internal semidirect product is internal quasi-central product, but the converse is not necessarily true.*

Proof. Suppose G is internal semidirect product of H and K. Thus, $G = HK$, $H \lhd G$ and $H \cap K = \{e\}$. Notice that $\{e\} \leq Z(G)$ and as such $H \cap K = \{e\}$. Therefore $G = H\tilde{o}K$.

The counter example in section 1 disprove the converse. □

Corollary 2.5. *Every internal direct product is internal quasi-central product, but the converse need not to be true.*

Proof. Suppose G is the internal direct product of H and K. Thus, $G = HK$, $H \lhd G$, $K \lhd G$ and $H \cap K$. Notice that $\{e\} \leq Z(G)$, as such $H \cap K = \{e\}$. Therefore $G = H\tilde{o}K$.

The counter example in section 1 disprove the converse. □

3. EXTERNAL QUASI-CENTRAL PRODUCT OF GROUPS

It is natural to ask similarly the definition of the external quasi-central product of two groups. In this section we give the definition of the external quasi-central product and we show that it is a group. Moreover we show that both internal and external quasi-central products are isomorphic. But before we begin our investigation, let us recall the definition of the external central product of two groups. If G_1 and G_2 are groups where A and B are the center of G_1 and G_2, respectively, such that there exists an isomorphism φ from A to B. Then the *external central product* of G_1 and G_2 is given by

$$G_1 o_\varphi G_2 = \{(g_1,\ g_2)N : g_1 \in G_1, g_2 \in G_2\},$$

where $N = \{(a,\ \phi(a^{-1})) :\ a \in A\}$ with operation defined by

$$(g_1,\ g_2)N(g_1',\ g_2')N = (g_1 g_1',\ g_2 g_2')N,$$

for $(g_1,\ g_2)N,\ (g_1',\ g_2')N \in G_1 o_\varphi G_2$. It is worth noting that N is a normal subgroup of the direct product of A and B.

The next definition gives the definition of the external quasi-central product of groups.

Definition 3.1. Let G_1 and G_2 be groups, $A \leq Z(G_1)$ and $B \leq Z(G_2)$ be such that there exists an isomorphism $\varphi : A \to B$. Let $\phi = \pi o \tau$ where $\tau : G_2 \to G_2/B$ (defined by $\tau(g_2) = g_2 B$, for $g_2 \in G_2$) and $\pi : G_2/B \to Aut(G_1)$ are homomorphisms. The *external quasi-central product* of G_1 and G_2 is defined by

$$\{(g_1,\ g_2)N : g_1 \in G_1, g_2 \in G_2\},$$

where $N = \{(a,\ \varphi(a^{-1})) : a \in A\}$, with operation defined by

$$(g_1,\ g_2)N(g_1',\ g_2')N = (g_1\phi_{g_2}(g_1'),\ g_2 g_2')N = (g_1'',\ g_2'')N,$$

where $g_1'' = g_1\phi_{g_2}(g_1') \in G_1$ and $g_2'' = g_2 g_2' \in G_2$.

We denote the external quasi-central product of two groups G_1 and G_2 by $G = G_1 \tilde{o}_\phi G_2$.

Remarks 1. *Since ϕ is a homomorphism of G_2 into $Aut(G_1)$, then $\phi_{g_2}(\phi_{g_2'}(g_1)) = \phi_{g_2 g_2'}(g_1)$, for $g_1 \in G_1$ and $g_2, g_2' \in G_2$.*

Theorem 3.2. *External quasi-central product of two groups is a group.*

Proof. Let G_1 and G_2 be groups with $A \leq Z(G_1)$ and $B \leq Z(G_2)$ such that there exists an isomorphism $\varphi : A \to B$ and let $N = \{(a, \varphi(a^{-1})) : a \in A \leq Z(G_1)\}$. Now, we show $G = G_1 \tilde{o}_\phi G_2$ is a group.

It is very clear from the definition of the operation that G is closed and associative.

Notice that for $(g_1,\ g_2)N \in G$, $(g_1, g_2)NN = (g_1,\ g_2)N = N(g_1,\ g_2)$ and notice also that e and f are identities in G_1 and G_2, respectively, as such $(e, f)N = N$ is the identity element in G.

Now let g_1^{-1} be inverse of $g_1 \in G$, g_2^{-1} be inverse of $g_2 \in G_2$, $e \in G_1$ and $f \in G_2$ be identities. Then notice that for $(g_1,\ g_2)N \in G$, we have:

$$\begin{aligned}(g_1,\ g_2)N(g_1,\ g_2)^{-1}N =&(g_1,\ g_2)N(\phi_{g_2^{-1}}(g_1^{-1}),\ g_2^{-1})N\\ =&(g_1\phi_{g_2}(\phi_{g_2^{-1}}(g_1^{-1})),\ g_2g_2^{-1})N\\ =&(g_1\phi_{g_2g_2^{-1}}(g_1^{-1}), f)N = (g_1g_1^{-1},\ f)N\\ =&(e,\ f)N = N.\end{aligned}$$

Moreover,

$$\begin{aligned}(g_1,\ g_2)^{-1}N(g_1,\ g_2)N =&(\phi_{g_2^{-1}}(g_1^{-1}),\ g_2^{-1})N(g_1,\ g_2)N\\ =&(\phi_{g_2^{-1}}(g_1^{-1}(\phi_{g_2^{-1}}g_1)),\ g_2^{-1}g_2)N\\ =&(\phi_{g_2^{-1}}(g_1^{-1}g_1),\ f)N = (\phi_{g_2^{-1}}(e),\ f)N\\ =&(e,\ f)N = N.\end{aligned}$$

This means that $(\phi_{g_2^{-1}}(g_1^{-1}),\ g_2^{-1})N$ is the inverse of $(g_1,\ g_2)N$ in G. Thus, every element in G has an inverse in G. Hence external quasi-central product of groups is a group. □

Corollary 3.3. *Every external central product is external quasi-central product, but the converse may not be true.*

Proof. Let G be external quasi central product of G_1 and G_2, then there exists an isomorphism $\varphi : A \to B$, where $A \leq Z(G_1)$ and $B \leq Z(G_2)$ and $N = \{(a, \varphi(a^{-1})) : a \in A\}$. Take $\tau : G_2 \to G_2/B$ to be natural homomorphism, $\pi : G_2/B \to \mathrm{Aut}(G_1)$ to be trivial homomorphism and $\phi = \pi o \tau$, then by Definition 3.1 $G = G_1\tilde{o}_\phi G_2$. The converse fails, see Example 3.6(d). □

Corollary 3.4. *Every external semi-direct product is external quasi-central product, but the converse may not be true.*

Proof. Let G be external semi-direct product of the groups G_1 and G_2, with e and f as identities of G_1 and G_2 respectively, then there exists a homomorphism $\phi : G_2 \to \mathrm{Aut}(G_1)$. Take $A = \{e\} \leq Z(G_1)$, $B = \{f\} \leq Z(G_2)$ and $N = \{(e, f)\}$, then by Definition 3.1 we have

$$G = \{(g_1, g_2)N : g_1 \in G_1, g_2 \in G_2\} = G_1\tilde{o}_\phi G_2,$$

as required.

The converse fails, see Example 3.6(d). □

Corollary 3.5. *Every external direct product is external quasi-central product, but the converse is not necessarily true.*

Proof. From Definition 3.1, take $B = \{f\}$, and let ϕ be a trivial automorphism then it follows that $G_1 \times G_2$ is $G_1 \tilde{o}_\phi G_2$.

Example 3.6(d) disprove the converse. □

Now let us demonstrate with some example how we obtain external quasi-central product of two groups.

Example 3.6. Let $G_1 = \langle a : a^4 = e \rangle$, $G_2 = \langle b : b^4 = f \rangle$.

(a) $G_1 \times G_2 = \{(e,f),(e,b),(e,b^2),(e,b^3),(a,f),(a,b),(a,b^2),(a,b^3),(a^2,f),(a^2,b),(a^2,b^2),(a^2,b^3),(a^3,f),(a^3,b),(a^3,b^2),(a^3,b^3)\}$, where $(g_1,g_2)(g_1',g_2') = (g_1g_1',g_2g_2')$ for g_1, $g_1' \in G_1$, and g_2, $g_2' \in G_2$ (see appendix A1 for the multiplication table of $G_1 \times G_2$).

It is easy to see from the multiplication table of $G_1 \times G_2$ (see appendix A1) that, (a,f) and (e,b) are of order 4 and that $(a,f)(e,b) = (e,b)(a,f) = (a,b)$. More so, for all $(x,y) \in G_1 \times G_2$, $(x,y) = (a^n,f)(e,b^m)$ for some $1 \leq m,n \leq 4$. Therefore, (a,f) and (e,b) generates $G_1 \times G_2$.

Hence, $G_1 \times G_2 = \langle (a,f),(e,b) : (a,f)^4 = (e,b)^4 = (e,f)$ and $(a,f)(e,b) = (e,b)(a,f) \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.

(b) Let $\varphi : G_2 \to \mathrm{Aut}(G_1)$ be homomorphism. Then $G_1 \rtimes_\varphi G_2 = \{(e,f),(e,b),(e,b^2),(e,b^3),(a,f),(a,b),(a,b^2),(a,b^3),(a^2,f),(a^2,b),(a^2,b^2),(a^2,b^3),(a^3,f),(a^3,b),(a^3,b^2),(a^3,b^3)\}$, where $(g_1,g_2)(g_1',g_2') = (g_1\varphi_{g_2}(g_1'),g_2g_2')$ for g_1, $g_1' \in G_1$, g_2, $g_2' \in G_2$ (see appendix A2 for the multiplication table of $G_1 \rtimes_\varphi G_2$).

One can see from the multiplication table of $G_1 \rtimes_\varphi G_2$ (see appendix A2) that, $(a,f)^2 = (a^2,f)$, $(a,f)^3 = (a^3,f)$, $(a,f)^4 = (e,f)$ and $(a,f)(a^3,f) = (a^3,f)(a,f) = (e,f)$. More so, $(e,b)^2 = (e,b^2)$, $(e,b)^3 = (e,b^3)$, $(e,b)^4 = (e,f)$, $(e,b)(e,b^3) = (e,b^3)(e,b) = (e,f)$ and also $(e,b)(a,f) = (a^3,f)(e,b)$.

Hence, $G_1 \rtimes_\varphi G_2 = \langle (a,f),(e,b) : (a,f)^4 = (e,b)^4 = (e,f)$ and $(e,b)(a,f) = (a^3,f)(e,b) \rangle \cong \mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_4$.

(c) Now if we let $A = \{e,a^2\} \leq G_1$, $B = \{f,b^2\} \leq G_2$ and

$$\pi : A \to B \text{ be defined by } \pi(e) = f,\ \pi(a^2) = b^2,$$

and also let $H = \{(x,\pi(x^{-1})) : x \in A\} = \{(e,f),(a^2,b^2)\}$.

Thus,

$G_1 o G_2 = (G_1 \times G_2)/H = \{\{(e,f),(a^2,b^2)\},\{(a,f),(a^3,b^2)\},\{(a,b),(a^3,b^3)\},\{(a,b^2),(a^3,f)\},\{(a,b^3),(a^3,b)\},\{(a^2,b),(e,b^3)\},\{(e,b),(a^2,b^3)\},\{(a^2,b),(e,b^2)\}\}$,

where $(x_1,y_1)H(x_2,y_2)H = (x_1x_2,y_1y_2)H$ $(x_1,x_2 \in G_1,\ y_1,y_2 \in G_2)$.

Now, let $E = \{(e,f),(a^2,b^2)\}$, $\beta_1 = \{(a,f),(a^3,b^2)\}$, $\beta_2 = \{(a,b),(a^3,b^3)\}$, $\beta_3 = \{(a,b^2),(a^3,f)\}$, $\beta_4 = \{(a,b^3),(a^3,b)\}$, $\beta_5 = \{(a^2,b),(e,b^3)\}$, $\beta_6 = \{(e,b),(a^2,b^3)\}$ and $\beta_7 = \{(a^2,b),(e,b^2)\}$.

The multiplication table of $G_1 o G_2$ is as follows.

E	β_1	β_2	β_3	β_4	β_5	β_6	β_7
β_1	β_7	β_5	E	β_6	β_4	β_2	β_3
β_2	β_5	E	β_6	β_7	β_1	β_3	β_4
β_3	E	β_6	β_7	β_5	β_2	β_4	β_1
β_4	β_6	β_7	β_5	E	β_3	β_1	β_2
β_5	β_4	β_1	β_2	β_3	β_7	E	β_6
β_6	β_2	β_3	β_4	β_1	E	β_7	β_5
β_7	β_3	β_4	β_1	β_2	β_6	β_5	E

It is easy to see from the above table that: $\beta_7 = \beta_1^2$, $\beta_3 = \beta_1^3$, $\beta_1^4 = \beta_4^2 = E$, $\beta_6 = \beta_1\beta_4$, $\beta_2 = \beta_1^2\beta_4$ and $\beta_1\beta_7 = \beta_7\beta_1$.

Hence, $G_1 o G_2 = \langle \beta_1, \beta_4 : \beta_1^4 = \beta_4^2 = E$ and $\beta_1\beta_4 = \beta_4\beta_1 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

(d) Finally, let A, B, π and H be as defined in (3.6) and let $\varphi : G_2 \to \mathrm{Aut}(G_1)$ be homomorphism.

Then $G_1 \tilde{o}_\varphi G_2 = (G_1 \rtimes_\varphi G_2)/H$
$= \{\{(e,f),(a^2,b^2)\},\{(a,f),(a^3,b^2)\},$
$\{(a,b),(a^3,b^3)\},\{(a,b^2),(a^3,f)\},\{(a,b^3),(a^3,b)\},\{(a^2,b),(e,b^3)\},$
$\{(e,b),(a^2,b^3)\},\{(a^2,b),(e,b^2)\}\}$, where $(x_1,y_1)H(x_2,y_2)H = (x_1\varphi_{y_1}(x_2), y_1y_2)H$, for all $x_1,x_2 \in G_1$, $y_1,y_2 \in G_2$. Now, let $E = \{(e,f),(a^2,b^2)\}$, $\beta_1 = \{(a,f),(a^3,b^2)\}$, $\beta_2 = \{(a,b),(a^3,b^3)\}$, $\beta_3 = \{(a,b^2),(a^3,f)\}$, $\beta_4 = \{(a,b^3),(a^3,b)\}$, $\beta_5 = \{(a^2,b),(e,b^3)\}$, $\beta_6 = \{(e,b),(a^2,b^3)\}$ and $\beta_7 = \{(a^2,b),(e,b^2)\}$.

The multiplication table of $G_1 \tilde{o}_\varphi G_2$ is as follows:

E	β_1	β_2	β_3	β_4	β_5	β_6	β_7
β_1	β_7	β_5	E	β_6	β_4	β_2	β_3
β_2	β_6	β_7	β_5	E	β_1	β_3	β_4
β_3	E	β_6	β_7	β_5	β_2	β_4	β_1
β_4	β_5	E	β_6	β_7	β_3	β_1	β_2
β_5	β_2	β_3	β_4	β_1	β_7	E	β_6
β_6	β_4	β_1	β_2	β_3	E	β_7	β_5
β_7	β_3	β_4	β_1	β_2	β_6	β_5	E

Now one can easily see from the table above that: $\beta_7 = \beta_1^2$, $\beta_3 = \beta_1^3$, $\beta_1^4 = \beta_4^4 = E$, $\beta_7 = \beta_4^2$, $\beta_2 = \beta_4^3 = \beta_1^2\beta_4$, $\beta_6 = \beta_1\beta_4$, $\beta_5 = \beta_1^3\beta_4$ and $\beta_4\beta_1 = \beta_3\beta_4 = \beta_1^{-1}\beta_4$.

Hence, $G_1 \tilde{o}_\varphi G_2 = \langle \beta_1, \beta_4 : \beta_1^4 = E,\ \beta_4^2 = \beta_1^2,\ \beta_4\beta_1 = \beta_1^{-1}\beta_4 \rangle \cong Q_8$.

Remarks 2. • *From the above example, one can easily see that:*

i) If $G = G_1 \times G_2$ or $G = G_1 \rtimes_\varphi G_2$, then $|G| = |G_1||G_2|$.

ii) If $G = G_1 o_\varphi G_2$ *or* $G = G_1 \tilde{o}_\phi G_2$*, then* $|G| = |G_1||G_2|/|G_1 \cap G_2|$.

Now we are going to show that internal and external quasi-central product of groups are isomorphic.

Theorem 3.7. *Let G be internal quasi-central product of H and K with* $H \lhd G$ *or* $K \lhd G$*. Then G is isomorphic to external quasi-central product of groups* H' *and* K'*, where* $H \cong H'$ *and* $K \cong K'$.

Proof. Let G be the internal quasi-central product of H and K, $T = (H' \rtimes_\phi K')/N$, where $N = \{(a, \pi(a^{-1})) : a \in A \leq Z(H')\}$ and $\pi : A \to B \leq Z(K')$ is an isomorphism. Define a mapping $\varphi : G \to T$ by

$$\varphi(g) = \varphi(hk) = (h', k')N$$

where $g = hk \in G$. Notice that $(h_1k_1) = (h_2k_2)$ if and only if $e = (h_1k_1)^{-1}(h_2k_2)$ if and only if $e = ((k_1^{-1}h_1^{-1}k_1)k_1^{-1})(h_2k_2)$ if and only if

$$e = ((k_1^{-1}h_1^{-1}k_1)(k_1^{-1}h_2k_1)k_1^{-1}k_2)$$

if and only if

$$\varphi(e) = \varphi((k_1^{-1}h_1^{-1}k_1)(k_1^{-1}h_2k_1)k_1^{-1}k_2)$$

if and only if

$$N = ((\phi_{k_1'^{-1}} h_1'^{-1} \phi_{k_1'^{-1}} h_2'), k_1'^{-1}k_2')N$$

if and only if $N = ((h_1', k_1')^{-1}(h_2', k_2'))N$ if and only if $(h_1', k_1')N = (h_2', k_2')N$ if and only if $\varphi(h_1k_1) = \varphi(h_2k_2)$. Therefore φ is well defined and injective.

Moreover, for any $(h', k')N \in T$, there exists $g = hk \in G$ such that $\varphi(g) = (h', k')N$. Therefore φ is surjective. Furthermore, given $g_1, g_2 \in G$, notice that $\varphi(g_1g_2) = \varphi(h_1k_1h_2k_2) = \varphi(h_1(k_1h_2k_1^{-1})k_1k_2) = (h_1'\phi_{k_1'}h_2', k_1'k_2')N = (h_1', k_1')N(h_2', k_2')N = \varphi(g_1)\varphi(g_2)$. Therefore φ is a homomorphism, hence φ is an isomorphism, as required. □

Corollary 3.8. *(see [6, Corollary 3.2.5]) Suppose G is a group, H and K are subgroups of G with H normal,* $G = HK = KH$ *and* $H \cap K = \{e\}$*, then there exits a homomorphism* $\varphi : K \to Aut(H)$ *such that G is isomorphic to the semi direct product of* $H \rtimes_\varphi K$.

Proof. The result follows from Corollary 2.4 and Theorem 3.7. □

Theorem 3.9. *If G is external quasi central product of* G_1 *and* G_2*, then G has normal subgroup isomorphic to* G_1 *and a subgroup isomorphic to* G_2.

Proof. Let $G = G_1 \tilde{\circ}_\phi G_2$ with e and f as identity elements in G_1 and G_2 respectively. Then for $g \in G$, $g = (g_1, g_2)N$, where $g_1 \in G_1, g_2 \in G_2$ and $N = \{(a, \varphi(a^{-1})) : a \in A = Z(G_1)\}$ with $\varphi : A \to B = Z(G_2)$ an isomorphism.

Now, let $G_1' = \{(g_1, f)N : g_1 \in G_1\}$. Notice that $e \in G_1$ and as such $(e, f)N \in G_1'$. Therefore $G_1' \neq \emptyset$ and that for $(g_1, f)N$, $(g_1', f)N \in G_1'$, it is easy to see that $(g_1, f)N(g_1', f)^{-1}N = (g_1'', f)N$, where $g_1'' = g_1(g_1')^{-1} \in G_1'$. That is to say $(g_1, f)N(g_1', f)^{-1}N \in G_1'$.

Moreover, for $(g_1, g_2)N \in G$, $(g_1', f)N \in G_1'$, one can also easily see that

$$\begin{aligned}(g_1, g_2)N(g_1', f)N(g_1, g_2)^{-1}N &= (g_1, g_2)(g_1', f)(\phi_{g_2^{-1}}(g_1^{-1}), g_2^{-1})N \\ &= (g_1'', f)N,\end{aligned}$$

where $g_1'' = g_1\phi_{g_2}(g_1' g_1^{-1}) \in G_1$.

Thus, $(g_1, g_2)N(g_1', f)N(g_1, g_2)^{-1}N \in G_1'$, which means that $G_1' \lhd G$.

Next, defined $\theta : G_1' \to G_1$ by $\theta((g_1, f)N) = g_1$, then it is easy to see that, θ is an isomorphism, as required.

Now, suppose $G_2' = \{(e, g_2)N : g_2 \in G_2\}$. In a similar fashion one can show that G_2' is a subgroup of G which is isomorphic to G_2. □

4. CONCLUDING REMARKS

We have successfully came up with a new product of groups called *quasi-central product*. We have shown that every central product is quasi-central product but not vice versa. Moreover, we defined both external and internal quasi-central products and further show that the external and internal quasi-central products are isomorphic.

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[1,2,3]Department of Mathematics, Bayero University Kano, P. M. B. 3011, Kano, Nigeria
E-mail addresses:
[1]aiguda.mth@buk.edu.ng, [2]mmzubairu.mth@buk.edu.ng, [3]aikiri01@yahoo.com
[4]Department of Mathematics, Kano State College of Education and Preliminary Studies, Kano, Nigeria
E-mail address: mbmadigawa@gmail.com

TABLE 1. Appendix A1

(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)
(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)
(e,b^2)	(e,b^3)	(e,f)	(e,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)
(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)
(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)
(a,b)	(a,b^2)	(a,b^3)	(a,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)
(a,b^2)	(a,b^3)	(a,f)	(a,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(e,b^2)	(e,b^3)	$(e.f)$	(e,b)
(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)
(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)
(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)
(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(e,b^2)	(e,b^3)	(e,f)	(e,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)
(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)
(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)
(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)
(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)	(e,b^2)	(e,b^3)	(e,f)	(e,b)
(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)

TABLE 2. Appendix A2

(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)
(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)
(e,b^2)	(e,b^3)	(e,f)	(e,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)
(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)
(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)
(a,b)	(a,b^2)	(a,b^3)	(a,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)
(a,b^2)	(a,b^3)	(a,f)	(a,b)	(e,b^2)	(e,b^3)	(e,f)	(e,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^2,b^2)	(a^2,b^3)	$(a^2.f)$	(a^2,b)
(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)
(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)
(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)
(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)	(e,b^2)	(e,b^3)	(e,f)	(e,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)
(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)
(a^3,f)	(a^3,b)	(a^3,b^2)	(a^3,b^3)	(e,f)	(e,b)	(e,b^2)	(e,b^3)	(a,f)	(a,b)	(a,b^2)	(a,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)
(a^3,b)	(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^2,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a,b)	(a,b^2)	(a,b^3)	(a,f)	(e,b)	(e,b^2)	(e,b^3)	(e,f)
(a^3,b^2)	(a^3,b^3)	(a^3,f)	(a^3,b)	(a^2,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a,b^2)	(a,b^3)	(a,f)	(a,b)	(e,b^2)	(e,b^3)	(e,f)	(e,b)
(a^3,b^3)	(a^3,f)	(a^3,b)	(a^3,b^2)	(e,b^3)	(e,f)	(e,b)	(e,b^2)	(a,b^3)	(a,f)	(a,b)	(a,b^2)	(a^2,b^3)	(a^2,f)	(a^2,b)	(a^2,b^2)