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ON SUBGROUPS OF A CLASS OF FINITE MINIMAL NONABELIAN 3-GROUP

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ABSTRACT. In this paper, we determined the number of subgroups of a finite nonabelian 3-group G defined by a presentation $\rho_1 \in G = \left\{a, b \mid a^9 = b^9 = e, [a,b] = a^3\right\}$, where a, b are generators of the same order. The form and the order of elements of the presentation group were obtained. We also drew the diagram of subgroups lattice and derived an explicit formula for counting the number of subgroups of the group.

1. INTRODUCTION

Classical set theory was introduced by George Cantor in the 1870s where it is either an element definitely belongs to a set or not. Berkovick [1] studied finite p-groups with few minimal nonabelian subgroups. Murali and Makamba [14] determined the number of subgroups of an abelian group of finite order $p^n q^m$ where p and q are different primes and Gautami [4] obtained a general formula for counting the number of subgroups of a p-group of arbitrary rank by using combinatorial arguments.

Calugaceanu [2] used Goursat's lemma for groups and derived an explicit formulae to find the number of subgroups of a finite

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abelian group. Also, Tarnauceanu [18] also used the arithmetic method to count the number of subgroups of a finite abelan group.

Petrillo [16] also used Goursat's lemma for groups to establish an explicit formula for finding the number of subgroups of the direct products of cyclic groups. Humera and Raza [7] determined the number of subgroups of finite nonabelian group defined by a presentation $G = \{x, y \mid x^2 = y^{16} = 1, yx = xy^7\}$. Laszlo [11] used simple number-theoretic arguments formulae to determine the number of cyclic subgroups of a finite abelian group and Hampejs and Laszlo [6] studied the number of subgroups of finite abelian groups of rank three by using the simple group-theoretic and number-theoretic arguments.

Sehgal and Yogesh [17] worked on the number of subgroup of finite abelian group $Z_m \bigotimes Z_n$. Johnson [9] and Kuku [10] discussed some basic concepts extensively.

EniOluwafe [3] also used the concept of fundamental group lattice to count the number of subgroups of finite nonmetacyclic 2-groups having no elementary abelian subgroup of order 8. Hampejs, Holighaus, Laszlo and Viesmeyr [5] established a formula for counting the number of subgroups of the group $Z_m \times Z_n$ by using simple group-theoretic and number theoretic arguments. Laszlo [12] deduced an explicit formulae for counting the subgroup of the group of the group $Z_m \times Z_n$ with the use of Goursat's lemma for groups. Olapade and EniOluwafe [15] computed a recurrent formula for counting the number of subgroups of $D_{2^n} \times C_2$ and Tarnauceanu and Laszlo [19] determined the number of subgroups of a given exponent in a finite abelian p-groups of rank two and of rank three. Iswariya and Rishivarman worked on arithmetic technique for nonabelian groups cryptosystem.

In this paper, we determine all subgroups of a finite nonabelian 3-group G defined by a presentation $\rho_1 \in G = \{a, b \mid a^9 = b^9 = e, [a, b] = a^3\}$, where a, b are generators of the same order and also draw the subgroups lattice of ρ_1. In section 2, we present some basic definitions and an example. In section 3, we find the order of each element, subgroups and obtain an explicit formula for counting the number of subgroups of ρ_1.

2. Preliminaries

This section defines some basic terms related to classical groups theory.

Definition 2.1. Johnson [9] The lattice of subgroups of a group G is the lattice whose elements are the subgroups of G, with the partial order relation being set inclusion. In this lattice, the join of two subgroups is the subgroup generated by their union, and the meet of two subgroups is the subgroup generated by their intersection.

Definition 2.2. Kuku [10] A $p-$group is a group in which every element has order equal to a power of p, where p is a prime number. Thus, a finite group is a $p-$group if its order is a power of p. Let the order of a nontrivial finite group G be $n = \prod_{i=1}^{t} p_i^{n_i}$. A subgroup $H \subseteq G$ is a p_i-group if its order is some power of p_i.

Definition 2.3. Kuku [10] Let G be a finite group and let p^β be the maximal power of prime p dividing $\mid G \mid$. Then

(i) every $p-$subgroup of G belongs to some subgroup of order p^β, hence there exists a sylow $p-$subgroup of G of order p^β
(ii) the number of sylow $p-$subgroups $n_p \cong 1(mod p)$
(iii) any two sylow $p-$subgroups are conjugate in G.

Definition 2.4. Humera and Raza [7] If G is finite group and H is a subgroup of G. Then, the order of H divides the order of G.

Definition 2.5. Johnson [9] A presentation of a group G is a description of the structure of G in terms of generators X and relations R. We can simply say that G has presentation $\{X \mid R\}$. Set X of generators is formed in a way such that every element of the group can be written as a product of powers of some of these generators and a set R of relations among those generators.

Example 2.6.

(1) The cyclic group of order n has the presentation $\{a \mid a^n = 1\}$ or $\{a \mid a^n\}$
(2) The dihedral group of order 6 could be defined as the group with generating set $X = \{x, y\}$ and relations $R = \left\{x^3 = 1, y^2 = 1, yxy = x^{-1}\right\}$. This may be written as $D_6 = \left\{x, y \mid x^3 = 1, y^2 = 1, yxy = x^{-1}\right\}$.

Definition 2.7. Johnson [9] The dihedral group of degree n and order $2n$ (for some n) is denoted by D_{2n} or D_{2^n}. It has the presentation

$$D_{2n} = \left\{a, b \mid a^n = b^2 = 1, a[a,b] = a^{-1}\right\},$$

for natural number $n \geq 2$. Thus
$D_{2n} = \{1, a, a^2, \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$

Example 2.8. Let $G = D_8$, dihedral group of order 8
$D_8 = \left\{a, b \mid a^4 = b^2 = 1, ab = ba^3\right\}$.
$D_8 = \left\{1, a, a^2, a^3, b, ab, a^2b, a^3b\right\}, n = 4$.
Let O represent the order of the element in D_8, then $O(b) = O(a^2) = O(ab) = O(a^2b) = O(a^3b) = 2$ and $O(a) = O(a^3) = 4$.

Subgroups of D_8**.** $\langle 1 \rangle$, $\left\langle a^2 \right\rangle = \{1, a^2\}$, $\langle b \rangle = \{1, b\}$, $\langle ab \rangle = \{1, ab\}$
$\langle a^2b \rangle = \{1, a^2b\}$, $\langle a^3b \rangle = \{1, a^3b\}$, $\langle a \rangle = \{1, a, a^2, a^3\} = \langle a^3 \rangle$,
$\langle a^2, b \rangle = \{1, a^2, b, a^2b\}$, $\langle a^2, ab \rangle = \{1, a^2, ab, a^3b\}$, $\langle a, b \rangle$.

Let O represent the order of the subgroup in D_8, then
$O(\langle a^2 \rangle) = O(\langle b \rangle) = O(\langle a^2b \rangle) = O(\langle ab \rangle) = O(\langle a^3b \rangle) = 2$ and $O(\langle a \rangle) = O(\langle a^2, b \rangle) = O(\langle a^2, ab \rangle) = 4$.

3. The Main Results

Let $G = \{a, b \mid a^{p^s} = b^{p^s} = e, [a,b] = a^{p^{(s-1)}}\}$ be a class of finite minimal nonabelian 3-group with generators of the same order, where a and b are the generators, $p = 3, s > 1$, of order p^{2s} and $[a,b] = a^{-1}b^{-1}ab$ is the commutator of elements a and b.

Example 3.1. Consider a presentation

$$\rho_1 \in G = \{a, b \mid a^9 = b^9 = e, [a,b] = a^3\}$$

of order 81 where $[a,b] = a^3$ implies that $ab = ba^4$. The elements of $\rho_1 \in G$ with generators a and b are written as
$e, a, a^2, \cdots, a^8, b, b^2, \cdots, b^8, ab, \cdots, ab^8, a^2b, \cdots, a^2b^8, a^3b, \cdots, a^3b^8, a^4b, \cdots, a^4b^8, a^5b, \cdots, a^5b^8, a^6b, \cdots, a^6b^8, a^7b, \cdots, a^7b^8, a^8b, \cdots, a^8b^8$.

TABLE 1. Summary of order of elements of ρ_1

Order	Elements
1	e
3	$a^3, a^6, b^3, b^6, a^3b^3, a^3b^6, a^6b^3, a^6b^6.$
9	all other 72 elements

3.1. **Summary of order of elements of $\rho_1 \in G$.** Table 1 summarizes the order of each element of the presentation group $\rho_1 \in G$. In this table, we are able to arrange the elements (formed by using the two generators a, b and the relations) of the presentation group ρ_1 of order 81 into classes of the same order. This is of great use for finding the lattice subgroups of the presentation group $\rho_1 \in G = \{a, b \mid a^9 = b^9 = e, [a, b] = a^3\}$.

Theorem 3.2. *The number of non trivial subgroups of $\rho_1 \in G$ of finite nonabelian group with generators of same order defined by a presentation*

$$\rho_1 \in G = \{a, b \mid a^9 = b^9 = e, [a, b] = a^3\}$$

is equal to 21.

3.2. **Order of subgroups.**

Order 3 subgroups. $\langle a^3 \rangle = \{e, a^3, a^6\}$, $\langle b^3 \rangle = \{e, b^3, b^6\}$
$\langle a^3b^6 \rangle = \{e, a^3b^6, a^6b^3\}$, $\langle a^3b^3 \rangle = \{e, a^3b^3, a^6b^6\}$

Order 9 subgroups. $\langle a^3, b^3 \rangle = \{e, a^3, a^6, b^3, b^6, a^3b^3, a^3b^6, a^6b^3, a^6b^6\}$
$\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$
$\langle ab^3 \rangle = \{e, ab^3, a^2b^6, a^3, a^4b^3, a^5b^6, a^6, a^7b^3, a^8b^6\}$
$\langle a^2b^3 \rangle = \{e, a^2b^3, a^4b^6, a^6, a^8b^3, ab^6, a^3, a^5b^3, a^7b^6\}$
$\langle b \rangle = \{e, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8\}$
$\langle a^6b \rangle = \{e, a^6b, a^3b^2, b^3, a^6b^4, a^3b^5, b^6, a^6b^7, a^3b^8\}$
$\langle a^3b \rangle = \{e, a^3b, a^6b^2, b^3, a^3b^4, a^6b^5, b^6, a^3b^7, a^6b^8\}$
$\langle a^7b \rangle = \{e, a^7b, a^2b^2, a^3b^3, ab^4, a^5b^5, a^6b^6, a^4b^7, a^8b^8\}$
$\langle ab \rangle = \{e, ab, a^8b^2, a^3b^3, a^4b^4, a^2b^5, a^6b^6, a^7b^7, a^5b^8\}$
$\langle a^4b \rangle = \{e, a^4b, a^5b^2, a^3b^3, a^7b^4, a^8b^5, a^6b^6, ab^7, a^2b^8\}$

TABLE 2. Multiplication table of elements of order 3 in ρ_1.

$\times$	e	a^3	a^6	b^3	b^6	a^3b^3	a^3b^6	a^6b^3	a^6b^6
e	e	a^3	a^6	b^3	b^6	a^3b^3	a^3b^6	a^6b^3	a^6b^6
a^3	a^3	a^6	e	a^3b^3	a^3b^6	a^6b^3	a^6b^6	b^3	b^6
a^6	a^6	e	a^3	a^6b^3	a^6b^6	b^3	b^6	a^3b^3	a^3b^6
b^3	b^3	a^3b^3	a^6b^3	b^6	e	a^3b^6	a^3	a^6b^6	a^6
b^6	b^6	a^3b^6	a^6b^6	e	b^3	a^3	a^3b^3	a^6	a^6b^3
a^3b^3	a^3b^3	a^6b^3	b^3	a^3b^6	a^3	a^6b^6	a^6	b^6	e
a^3b^6	a^3b^6	a^6b^6	b^6	a^3	a^3b^3	a^6	a^6b^3	e	b^3
a^6b^3	a^6b^3	b^3	a^3b^3	a^6b^6	a^6	b^6	e	a^3b^6	a^3
a^6b^6	a^6b^6	b^6	a^3b^6	a^6	a^6b^3	e	b^3	a^3	a^3b^3

$\langle a^8b\rangle = \{e, a^8b, ab^2, a^6b^3, a^5b^4, a^7b^5, a^3b^6, a^2b^7, a^4b^8\}$
$\langle a^2b\rangle = \{e, a^2b, a^7b^2, a^6b^3, a^8b^4, a^4b^5, a^3b^6, a^5b^7, ab^8\}$
$\langle a^5b\rangle = \{e, a^5b, a^4b^2, a^6b^3, a^2b^4, ab^5, a^3b^6, a^8b^7, a^7b^8\}$.

Order 27 subgroups. $\langle a, b^3\rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, b^3, ab^3, a^2b^3, a^3b^3, a^4b^3, a^5b^3, a^6b^3, a^7b^3, a^8b^3, b^6, ab^6, a^2b^6, a^3b^6, a^4b^6, a^5b^6, a^6b^6, a^7b^6, a^8b^6\}$
$\langle b, a^3\rangle = \{e, b, b^2b^3, b^4, b^5, b^6, b^7, b^8, a^3, a^3b, a^3b^2, a^3b^3, a^3b^4, a^3b^5, a^3b^6, a^3b^7, a^3b^8, a^6, a^6b, a^6b^2, a^6b^3, a^6b^4, a^6b^5, a^6b^6, a^6b^7, a^6b^8\}$
$\langle a^7b, a^3\rangle = \{e, a^7b, a^2b^2, a^3b^3, ab^4, a^5b^5, a^6b^6, a^4b^7, a^8b^8, a^3, ab, a^5b^2, a^6b^3, a^4b^4, a^8b^5, b^6, a^7b^7, a^2b^8, a^6, a^4b, a^8b^2, b^3, a^7b^4, a^2b^5, a^3b^6, ab^7, a^5b^8\}$
$\langle a^2b, a^3\rangle = \{e, a^2b, a^7b^2, a^6b^3, a^8b^4, a^4b^5, a^3b^6, a^5b^7, ab^8, a^3, a^5b, ab^2, b^3, a^2b^4, a^7b^5, a^6b^6, a^8b^7, a^4b^8, a^6, a^8b, a^4b^2, a^3b^3, a^5b^4, ab^5, b^6, a^2b^7, a^7b^8\}$
$\langle a, b\rangle = \rho_1 \in G$

Table 2 illustrates how to generate subgroups of the presentation group $\rho_1 \in G$ by multiplying the identity element e together with other elements of order 3 (that is, elements $a^3, a^6, b^3, b^6a^3b^3, a^3b^6, a^6b^3, a^6b^6$ as described in the table above, where $\rho_1 = \{a, b \mid a^9 = b^9 = e, [a,b] = a^3\}$.

Theorem 3.3. *If $G = \{a, b \mid a^{p^s} = b^{p^s} = e, [a,b] = a^{p^{(s-1)}}\}$ is a class of finite minimal nonabelian $p-$groups with $s > 1$ and $p = 3$. Then the* *total number of subgroups of G is given by the expression* $\sum_{i=1}^{s+1}[(2s-2i)+3]p^{i-1}$.

Remark 3.4. The expression $\sum_{i=1}^{s+1}(2s-2i+3)p^{i-1}$ can be simplified as $2\left\{\frac{s(p^{s-1}-1)}{p-1}-\frac{(p^{s-1}-1)-(s-1)(p-1)p^{s-1}}{(p-1)(1-p)}\right\}-2p^{s}+\frac{3(p^{s+1}-1)}{p-1}$. The equality can be proved by induction on s.

Proof. We prove by induction on s.

$$\sum_{i=1}^{s+1}[(2s-2i)+3]p^{i-1}=2\left\{s\sum_{i=1}^{s-1}p^{i-1}-\sum_{i=1}^{s-1}ip^{i-1}\right\}-2p^{s}+3\sum_{i=1}^{s+1}p^{i-1}, \tag{3.1}$$

where (i)

$$s\sum_{i=1}^{s-1}p^{i-1}=\frac{s(p^{s-1}-1)}{p-1}$$

(ii)

$$\sum_{i=1}^{s-1}ip^{i-1}=\frac{(p^{s-1}-1)-(s-1)(p-1)p^{s-1}}{(p-1)(1-p)}$$

(iii)

$$\sum_{i=1}^{s+1}p^{i-1}=\frac{(p^{s+1}-1)}{p-1}$$

Now substitute (i), (ii) and (iii) into 3.1, the RHS becomes

$$2\left\{\frac{s(p^{s-1}-1)}{p-1}-\frac{(p^{s-1}-1)-(s-1)(p-1)p^{s-1}}{(p-1)(1-p)}\right\}-2p^{s}+\frac{3(p^{s+1}-1)}{p-1}$$

For p = 3 and s = 2, the LHS becomes

$$\sum_{i=1}^{3}(4-2i+3)p^{i-1}=\sum_{i=1}^{3}(7-2i)p^{i-1}=5+3p+p^{2}=23$$

Also, the RHS becomes

$$2\left\{\frac{2(p-1)}{p-1}-\frac{(p-1)-(p-1)p}{(p-1)(1-p)}\right\}-2p^{2}+\frac{3(p^{3}-1)}{p-1}$$

$$= 2\left\{\frac{2(3-1)}{2} - \frac{(3-1)-(2)3}{(2)(-2)}\right\} - 2(3)^2 + \frac{3(3^3-1)}{2}$$

$$= 2\left\{\frac{2(2)}{2} - \frac{(2)-(2)3}{(-4)}\right\} - 2(3)^2 + \frac{3(26)}{2} = 23$$

To prove (i). Assuming it holds for some $s = k$,

$$k\sum_{i=1}^{k-1} p^{i-1} = \frac{k(p^{k-1}-1)}{p-1} \tag{3.2}$$

For $s = k+1$, we have

$$(k+1)\sum_{i=1}^{k} p^{i-1} = \frac{(k+1)(p^k-1)}{p-1} \tag{3.3}$$

Then LHS of 3.3 is written as

$$(k+1)\sum_{i=1}^{k} p^{i-1} = (k+1)\sum_{i=1}^{k-1} p^{i-1} + (k+1)p^{k-1} \tag{3.4}$$

Now substitute 3.2 in the RHS of 3.4, we obtain $\dfrac{(k+1)(p^{k-1}-1)}{p-1} +$ $(k+1)p^{k-1}$

$= \dfrac{(k+1)(p^{k-1}-1)+(k+1)(p-1)p^{k-1}}{p-1}$

$= \dfrac{(k+1)(p^{k-1}-1+(p-1)p^{k-1})}{p-1} = \dfrac{(k+1)(p^k-1)}{p-1}$

$(k+1)\sum_{i=1}^{k} p^{i-1} = \dfrac{(k+1)(p^k-1)}{p-1}$

Thus, LHS = RHS

To prove (ii). Assume it holds for $s = k$

$$\sum_{i=1}^{k-1} ip^{i-1} = \frac{(p^{k-1}-1)-(k-1)(p-1)p^{k-1}}{(p-1)(1-p)} \tag{3.5}$$

For $s = k+1$

$$\sum_{i=1}^{k} ip^{i-1} = \frac{(p^k-1)-k(p-1)p^k}{(p-1)(1-p)} \tag{3.6}$$

Then, LHS of 3.6 is written as

$$\sum_{i=1}^{k} ip^{i-1} = \sum_{i=1}^{k-1} ip^{i-1} + kp^{k-1} \tag{3.7}$$

Now substitute 3.5 in the RHS of 3.7, we obtain

$$\frac{(p^{k-1}-1)-(k-1)(p-1)p^{k-1}}{(p-1)(1-p)} + kp^{k-1}$$

$$= \frac{(p^{k-1}-1)-(k-1)(p-1)p^{k-1}+k(p-1)(1-p)p^{k-1}}{(p-1)(1-p)}$$

$$= \frac{-1+p^k+kp^k-kpp^k}{(p-1)(1-p)} = \frac{(p^k-1)-k(p-1)p^k}{(p-1)(1-p)}$$

Thus, LHS = RHS.
To prove (iii). Now assume it is true for s = k.

$$\sum_{i=1}^{k+1} p^{i-1} = \frac{(p^{k+1}-1)}{p-1} \tag{3.8}$$

Now, prove it is true for $s = k+1$

$$\sum_{i=1}^{k+2} p^{i-1} = \frac{(p^{(k+1)+1}-1)}{p-1} \tag{3.9}$$

The LHS of 3.9 is written as

$$\sum_{i=1}^{k+2} p^{i-1} = \sum_{i=1}^{k+1} p^{i-1} + p^{(k+2)-1} \tag{3.10}$$

Now substitute 3.8 in the RHS of 3.10, we get

$$\frac{(p^{k+1}-1)}{p-1} + p^{(k+2)-1} = \frac{(p^{k+1}-1)+(p-1)p^{k+1}}{p-1} = \frac{(p^{(k+1)+1}-1)}{p-1}$$

Thus LHS = RHS. Hence by induction on s, the statement is true for all primes and natural number $s > 1$. $\square$

TABLE 3. Number of subgroups of $\rho_1, \rho_2, \cdots, \rho_5$ in G.

order	ρ_1: p=3, s=2	ρ_2: p=3, s=3	ρ_3: p=3, s=4	ρ_4: p=3, s=5	ρ_5: p=3, s=6
1	1	1	1	1	1
3	4	4	4	4	4
9	13	13	13	13	13
27	4	40	40	40	40
81	1	13	121	121	121
243	•	4	40	364	364
729	•	1	13	121	1093
2187	•	•	4	40	364
6561	•	•	1	13	121
19683	•	•	•	4	40
59049	•	•	•	1	13
177147	•	•	•	•	4
531441	•	•	•	•	1
$\sum$	23	76	237	722	2179

This table summarizes the number of subgroups of the presentation groups $\rho_1, \rho_2, \cdots, \rho_5$. In this table, the order of each subgroup and the number of possible subgroups for different orders were indicated. The $\sum$ row shows the number of subgroups of presentation groups of order 81, 729, 6561, 59049,and 531441 respectively.

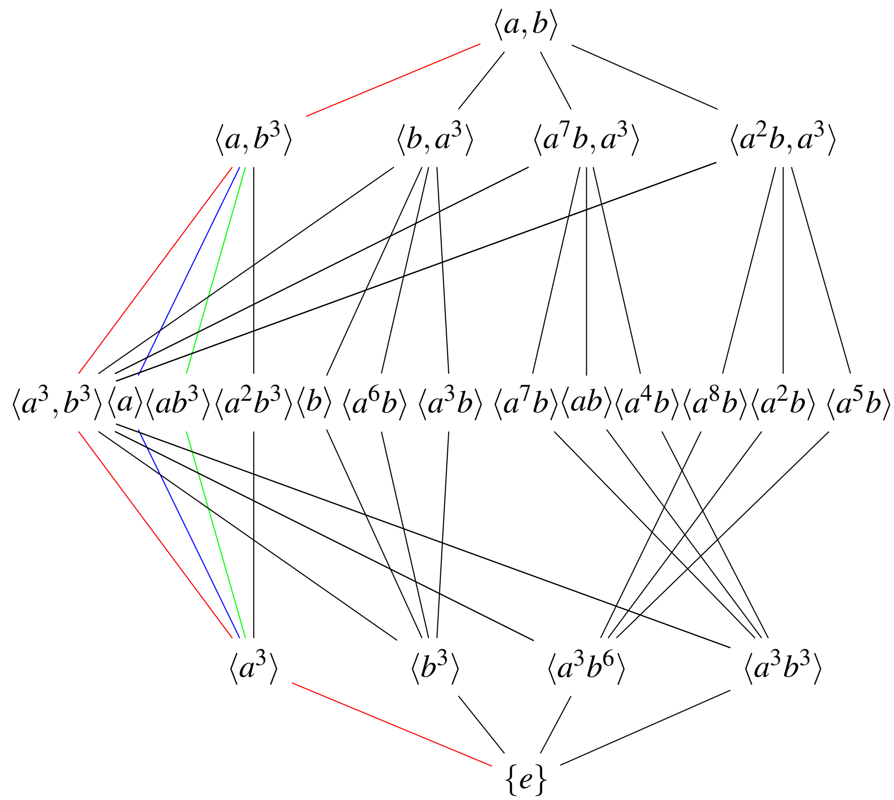

FIGURE 1. Hasse diagram showing subgroups of ρ_1

Figure 1 illustrates the diagram of the lattice subgroups of the presentation group $\rho_1 \in G$. In this figure we are able to show the generators of each subgroup according to the order of each generator from the identity element e to generator $\langle a,b\rangle = \rho_1 = \{a,b \mid a^9 = b^9 = e, [a,b] = a^3\}$.

4. CONCLUSION

In this paper, we determined the number of subgroups of a finite non-abelian 3-group G defined by a presentation $\rho_1 \in G = \left\{a,b \mid a^9 = b^9 = e, [a,b] = a^3\right\}$, where a,b are generators of the same order. The form and the order of elements of the presentation group were obtained. We also drew the diagram of subgroups lattice and derived an explicit formula for counting the number of subgroups of the group.

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