Journal of the Nigerian Mathematical Society
Vol. 38, Issue 3, pp. 569-592, 2019

Special Issue: The International Conference on Non-linear Analysis (organized to mark the 70th birthday anniversary of Prof. Charles Ejike Chidume)

APPROXIMATION OF SOLUTIONS OF GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINTS OF MULTIVALUED ASYMPTOTICALLY QUASINONEXPANSIVE MAPPINGS

BASHIR ALI[1] AND LAWAL UMAR

ABSTRACT. In this paper, we introduce an iterative method for finding a solution of mixed equilibrium problem which is a common fixed point of a countable family of multivalued asymptotically quasi nonexpansive mapping in a uniformly convex Banach space. Strong convergence theorems are proved under some conditions. Our results generalize and improve recent results announced by many authors.

Keywords and phrases: Variational inequality problems; fixed point problems; generalized mixed equilibrium problems; asymptotically quasi nonexpansive multivalued mappings.
2010 Mathematics Subject Classification: 47H09, 47J25.

1. INTRODUCTION

Let E be a real Banach space with norm $\| \,.\, \|$, E^* be the dual space of E and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \forall y \in C.$$

This problem was first studied by Blum and Oettli [4]. The set of solutions of equilibrium problem is denoted by $EP(f)$ that is $EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \forall y \in C\}$. Let $A : C \longrightarrow E^*$ be a nonlinear mapping. The variational inequality problem with respect to A and C is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. The

Received by the editors December 31, 2018; Revised August 19, 2019 ; Accepted: November 07, 2019
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

[1]Corresponding author

set of solutions of variational inequality problems with respect to C and A is denoted by $VI(C,A)$. Setting $f(x,y) = \langle Ax, y-x\rangle$ for all $x,y \in C$, then $\hat{x} \in EP(f)$ if and only if $\langle A\hat{x}, y-\hat{x}\rangle \geq 0$ for all $y \in C$, i.e $\hat{x}$ is a solution of the variational inequality with respect to A and C. Let $\varphi : C \longrightarrow \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semi continuous, then the minimization problem of φ is to find $\hat{x} \in C$ such that $\varphi(\hat{x}) \leq \varphi(y)\ \forall y \in C$.
The generalised equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y-\hat{x}\rangle \geq 0,\ \forall y \in C.$$

The set of solutions of generalised equilibrium problem is denoted by

$$GEP(f,A) := \{\hat{x} \in C : f(\hat{x}, y) + \langle A\hat{x}, y-\hat{x}\rangle \geq 0,\ \forall y \in C\}.$$

In this paper, we are interested in solving equilibrium problem with respect to f given by

$$f(x,y) = \sum_{i=1}^{k} f_i(x,y), \forall x,y \in C,$$

where $f_i : C \times C \longrightarrow \mathbb{R}$ are bifunctions for i =$1,2,3,...k$, satisfying the conditions (A_1) - (A_4) below;
A_1) $f_i(x,x) = 0$, for all $x \in C$, for $i = 1,2,3...k$;
A_2) f_i is monotone,i.e $f_i(x,y)+f_i(y,x) \leq 0$, for each $i \in \{1,2,3,...,k\}$
A_3) for all $x,y \in C$, we have $\limsup\limits_{t\to\infty} f_i(tz+(1-t)x, y) \leq f_i(x,y)$;
A_4) for all $x \in C, f_i(x,.)$ is convex and lower semi continuous $\forall i \in \{1,2,3,...,k\}$.

The mixed equilibrium problem is to find $\widehat{x} \in C$ such that

$$\sum_{i=1}^{k} f_i(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C.$$

The set of solution of mixed equilibrium problem is denoted by

$$MEP(f,\varphi) := \{\hat{x} \in C : \sum_{i=1}^{k} f_i(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C\}.$$

The generalised mixed equilibrium problem is to find $\widehat{x} \in C$ such that

$$\sum_{i=1}^{k} f_i(\hat{x}, y) + \langle A\hat{x}, y-\hat{x}\rangle + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C.$$

The set of solutions of generalised mixed equilibrium problem is denoted by

$$GMEP(f, A, \varphi) := \{\hat{x} \in C : \sum_{i=1}^{k} f_i(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C\}.$$

The generalized mixed equilibrium problems are suitable method for investigating various applied problems arising in economics, mathematical physics, engineering and other fields. Moreover equilibrium problems are closely related with other general problems in nonlinear analysis such as fixed point, game theory, variational inequality and optimization problems.
Let E be a Banach space and C be a nonempty subset of E. Let $\hat{C}B(C)$ and $K\hat{C}(C)$ denote the families of nonempty, closed and bounded subsets and nonempty compact and convex subsets of C respectively. The Hausdorff metric on $\hat{C}B(C)$ is defined by

$$H(A, B) = max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

for $A, B \in \hat{C}B(C)$, where $d(x, C) = \inf\{\|x - y\| : y \in C\}$, is the distance from a point x to a subset C. A multivalued mapping $T : C \longrightarrow \hat{C}B(C)$ is called nonexpansive if $H(Tx, Ty) \leq \|x-y\|$ for all $x, y \in C$. The set of all fixed points of the map $T : C \longrightarrow \hat{C}B(C)$ will be denoted by $F(T) = \{p \in C : p \in Tp\}$.

In 1969, Nadler [11] proved a fixed point theorem for multivalued contraction mappings. He extended theorems on the stability of fixed points of single-valued mapping and also gave a counter example to a theorem about (ε, λ)- uniformly locally expansive (single-valued) mappings. Later in 1997, Huang [9] studied approximation of common fixed point of two nonexpansive multivalued mappings satisfying certain contractive condition. In 2005, Sastry and Babu [14] proved convergence theorems for multivalued mappings using Ishikawa and Mann iterates. In 2007, a generalization of the results of Sastry and Babu [14] to uniformly convex Banach spaces was given by Panyanak [13]. He proved a convergence theorem for a mapping defined on a non compact domain. Later in 2008, Song and Wang [17] proved strong convergence of Mann and Ishikawa iterates to a fixed point of multivalued nonexpansive mapping T under some appropriate conditions. In 2009, Shahzad and Zegeye [16] proved strong convergence theorems for fixed point of quasi nonexpansive multivalued mapping using Ishikawa iterates. They proved their result without imposing the condition $Tp = \{p\}, \forall p \in F(T)$ and also relaxed the compactness condition of the domain of T. Abbas and Rhoades [1] established weak and strong convergence theorems for common fixed points of two multivalued nonexpansive mappings in a real uniformly convex Banach space under some basic boundary conditions.

In 2009, Takahashi and Zembayashi [18], introduced two iterative sequences, for finding a common element of the set of fixed points of a relatively nonexpansive mapping S and the set of solutions of an equilibrium problem in Banach space as follows :

$$\begin{cases} x_0 = x \in C \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \textit{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\} \\ x_{n+1} = \Pi_{H_n \cap W_n} x, n \geq 0, \end{cases}$$

$$\begin{cases} x_n \in C \textit{ such that } f(x_n, y) + \dfrac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \forall y \in C \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \ n \in \mathbb{N}, \end{cases}$$

where $\phi : E \times E \longrightarrow \mathbb{R}^+$ denotes the Lyapunov functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall y, x \in E.$$

J is the normalized duality mapping on E, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty]$ for some $a > 0$. They proved strong convergence of the scheme to a common element of the set of fixed points of relatively nonexpansive mapping and the set of solution of an equilibrium problem in a Banach space. Moreover they proved weak convergence of the scheme to the same point. In 2012, Bunyawat and Suantai [3] introduce an iteration method for finding a common fixed point of a countable family of multivalued quasi nonexpansive mappings in a uniformly convex Banach space. Deng et, al.[6], proved strong convergence theorem for common fixed point of finite family of asymptotically nonexpansive mappings and the set of solution of mixed equilibrium problem in uniformly smooth and uniformly convex Banach space. In 2016, Ezeora [7], proved strong convergence theorems for a common element of the set of solution of generalized mixed equilibrium problem

and the set of common fixed points of a finite family of multivalued strictly pseudocontractive mappings in real Hilbert spaces.
In this paper, motivated and inspired by the results mentioned above, we prove strong convergence theorems for finding a point in the set of common fixed point of a countable family of multivalued asymptotically quasi nonexpansive mappings and the sets of solutions of generalized mixed equilibrium problem in Banach space. Our results generalize and improve recent results announced by many authors.

Let $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi) \neq \emptyset$ and $x^* \in F$, generate a sequence $\{x_n\}$ defined by

$$\begin{cases} u_1 \in E \\ x_n \in C \textit{ such that } \displaystyle\sum_{j=1}^{k} f_j(x_n, y) + \langle Ax_n, y - x_n \rangle \\ +\varphi(y) - \varphi(x_n) + \dfrac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C \\ u_{n+1} = \alpha_{0,n} x_n + \displaystyle\sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n},\ w_{i,n} \in T_i^n x_n, n \geq 1, \end{cases} \tag{1}$$

where the sequences $\{\alpha_{i,n}\} \subset [0, 1]$ satisfy $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$, $\limsup_{n\to\infty} \alpha_{0,n} < 1$, $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, $w_{i,n} \in T_i^n x_n$ such that $T_i x^* = \{x^*\}$ for all $i \in \{1, 2, 3, ...\}$ and let $\{r_n\} \subset [a, \infty]$, for some $a > 0$, where T_i^n stand for the n compositions of T_i. Assume that the condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is satisfied.

The main purpose of this paper is to prove convergence of the scheme (1) to a point in F.

2. PRELIMINARY

Definition 1: A multivalued mapping $T : C \longrightarrow \hat{C}B(C)$ is said to be:
i) Quasi nonexpansive if $F(T) \neq \emptyset$, and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in C$ $p \in F(T)$
ii) Asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $H(T^n x, T^n y) \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \in N$.
iii) Asymptotically quasi nonexpansive if $F(T) \neq \emptyset$ and there exists

a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $H(T^n x, T^n p) \leq k_n\|x - p\|$, for all $x \in C$, $p \in F(T)$.

Definition 2: A multivalued mapping $T : C \longrightarrow \hat{C}B(C)$ is said to be hemi-compact, if for any sequence $\{x_n\} \subset C$ such that $d(x_n, Tx_n) \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p \in C$. We note that if C is compact, then every multivalued mapping T is hemi compact.

Remark 1:

(i) The class of quasi asymptotically nonexpansive multivalued mappings contains properly the class of quasi nonexpansive multivalued mappings with fixed points.

(ii) The class of quasi nonexpansive multivalued mappings contains properly the class of nonexpansive multivalued mappings with fixed points.

Definition 3: Let T be a multivalued mapping the nth composition of T denoted by $T^n = T(T^{n-1})$ is defined by

$$T^n x = \bigcup_{y \in T^{n-1}x} Ty.$$

Definition 4: An asymptotically multivalued quasi nonexpansive mapping T is said to be uniformly $L-$Lipshitzian if there exists $L > 0$ such that for any pair $x, y \in D(T)$ we have

$$H(T^n x, T^n y) \leq L\|x - y\|.$$

Definition 5: A multivalued mapping T is said to be continuous if whenever a sequence $\{x_n\} \subset D(T)$ is such that $x_n \to x$ as $n \to \infty$, then $Tx_n \to Tx$.

Throughout this paper, we denoted by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integer and real numbers, respectively. Let E be a Banach space and E^* be the dual of E, We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$ respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \rightharpoonup x^*$, for all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^*\rangle$, which is called duality pairing. The normalized duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. A Banach space E is said to be strictly convex if $\dfrac{\| x + y \|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The space E is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$,

there exists $\delta > 0$ such that $\dfrac{\| x+y \|}{2} \leq 1-\delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$. A Banach space is said to have Kadec-Klee property, if $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ imply $x_n \longrightarrow x$. Every Hilbert space and uniformly convex Banach space possess Kadec - Klee property. The space E is said to be smooth if the limit $\lim\limits_{t \to 0} \dfrac{\| x+ty \| - \| x \|}{t}$ exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$.
A Banach space E is said to satisfy *Opial's condition* [12] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ as $n \to \infty$ implies that

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \ \forall y \in E, y \neq x.$$

It is known that if E is smooth, strictly convex and reflexive, then the duality mapping J is single - valued, one-to-one and onto. The duality mapping J is said to be weakly sequentially continuous if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup Jx$.
Following Alber [2], the generalised projections Π_C from E onto C is defined by

$$\Pi_C(x) = \underset{y \in C}{argmin} \phi(y, x), \forall x \in E.$$

The generalised projection Π_C from E onto C is well defined, single valued and satisfies

$$(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|x\| + \|y\|)^2, \forall x, y \in E$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + \langle x - z, Jz - Jy \rangle.$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C becomes the metric projection of H onto C. The following lemmas for generalised projections are well known.

Lemma 1: (see [2]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, Then $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \forall x \in C$ and $y \in E$

Lemma 2: (see [2, 10]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space. Let $x \in E$ and $z \in C$. Then $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jy \rangle \leq 0, \forall y \in C$.
A mapping $T : C \longrightarrow C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if

there exists a sequence $\{x_n\}$ in C which converges weakly to p and $\lim\limits_{n\to\infty} \| x_n - Tx_n \| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Lemma 3: (see [8]) Let E be a uniformly convex Banach space satisfying the Opial's condition, C a nonempty closed subset of E and $T : C \longrightarrow C$ asymptotically nonexpansive mapping. If the sequence $\{x_n\} \subset C$ is a weakly convergent with the weak limit p and if $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tp = p$

Lemma 4: (see [15]) Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integer n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup\limits_{n\to\infty}\|x_n\| \leq r, \limsup\limits_{n\to\infty}\|y_n\| \leq r$, and $\lim\limits_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\limsup\limits_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 5: (see [5]) Let E be a uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{\|x \in E : \|x\| \leq r\}$. Then for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda\}_{n=1}^{\infty}$ of positive numbers such that $\sum\limits_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous strictly increasing convex function

$$g : [0, 2r] \longrightarrow \mathbb{R}, g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\| \sum_{n=1}^{\infty} \lambda_n x_n \|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

Lemma 6: (see [4]) Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$, let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that $f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0$, for all $y \in C$.

Lemma 7: (see [19]) Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be sequences of non-negative real numbers satisfying
(1) $a_{n+1} \leq a_n + b_n$
(2) $a_{n+1} \leq (1 + c_n)a_n + b_n$, for all $n \in \mathbb{N}$, where $\sum\limits_{n=1}^{\infty} b_n < \infty$ and

$\sum_{n=1}^{\infty} c_n < \infty$. Then

i) $\lim_{n\to\infty} a_n$ exists.

ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 8: (see [10]) Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing continuous and convex function $g : [0, 2r] \longrightarrow \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$ for all $x, y \in B_r$.

Lemma 9: Let $\varphi : C \longrightarrow \mathbb{R}$ be lower semi- continuous and convex function, $A : C \longrightarrow E^*$ be continuous and monotone and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ as follows:

(A_1) $f(x, x) = 0$

(A_2) f is monotone i.e, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$

(A_3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$

(A_4) the mapping $y \mapsto f(x, y)$ is convex and lower semi continuous.

Then $\tau : C \times C \longrightarrow \mathbb{R}$ defined by

$\tau(x, y) = \sum_{j=1}^{k} f_j(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A_1) - (A_4)$.

Proof:

$$(A_1) : \tau(u, u) = \sum_{j=1}^{k} f_j(u, u) + \langle Au, u - u\rangle + \varphi(u) - \varphi(u) = 0,$$

$$\begin{aligned}
(A_2) : \tau(u, y) + \tau(y, u) &= \sum_{j=1}^{k} f_j(u, y) + \sum_{j=1}^{k} f_j(y, u) \\
&+ \langle Au, y - u\rangle + \langle Ay, u - y\rangle \\
&+ \varphi(y) - \varphi(u) + \varphi(u) - \varphi(y) \\
&= \sum_{j=1}^{k} f_j(u, y) + \sum_{j=1}^{k} f_j(y, u) + \langle Ay - Au, u - y\rangle \\
&= \sum_{j=1}^{k} f_j(u, y) + \sum_{j=1}^{k} f_j(y, u) - \langle Ay - Au, y - u\rangle \\
&\leq 0
\end{aligned}$$

$$
\begin{aligned}
(A_3) : \limsup_{t\to 0}\tau(tz+(1-t)x,y) &= \limsup_{t\to 0}\Big(\sum_{j=1}^{k} f_j(tz\\
&+ (1-t)x,y) + \langle A(tz+(1-t)x, y-(tz+(1-t)x)\rangle\\
&+ \varphi(y)-\varphi(tz+(1-t)x))\Big)\\
&\le \limsup_{t\to 0}(\sum_{j=1}^{k} f_j(tz+(1-t)x,y)\\
&\quad + \limsup_{t\to 0}\langle A(tz+(1-t)x, y-(tz+(1-t)x)\rangle\\
&\quad + \limsup_{t\to 0}(\varphi(y)-\varphi(tz+(1-t)x))\\
&\quad \le \sum_{j=1}^{k} f_j(x,y) + \langle Ax, y-x\rangle + \varphi(y)-\varphi(x)\\
&\quad = \tau(x,y)
\end{aligned}
$$

$$
\begin{aligned}
(A_4) : \tau(u,tz+(1-t)y) &= \sum_{j=1}^{k} f_j(u,tz+(1-t)y)\\
&+ \langle Au, tz+(1-t)y-u\rangle + \varphi(tz+(1-t)y)-\varphi(u)\\
&\le t\sum_{j=1}^{k} f_j(u,z) + (1-t)\sum_{j=1}^{k} f_j(u,y)\\
&+ t\langle Au, z-u\rangle + (1-t)\langle Au, y-u\rangle\\
&+ t\varphi(z)+(1-t)\varphi(y)-t\varphi(u)-(1-t)\varphi(u)\\
&= t[\sum_{j=1}^{k} f_j(u,z) + \langle Au, z-u\rangle + \varphi(z)-\varphi(u)]\\
&+ (1-t)[\sum_{j=1}^{k} f_j(u,y) + \langle Au, y-u\rangle + \varphi(y)-\varphi(u)]\\
&= t\tau(u,z)+(1-t)\tau(u,y)
\end{aligned}
$$

Lemma 10: (see [20]) Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \longrightarrow E^*$ be a continuous and monotone mapping, $\zeta : C \longrightarrow \mathbb{R}$ be a lower semi- continuous and convex function, and $h : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions

$(A1)-(A4)$. Let $r>0$ be any given number and $u\in E$ be any given point. Then, the followings hold:
(1) There exists $z\in C$ such that

$$h(z,v)+\zeta(v)-\zeta(v)+\langle v-z,Az\rangle+\frac{1}{r}\langle v-z,Jz-Ju\rangle\geq 0,\forall v\in C.$$

(2) If we define a mapping $A_r:E\longrightarrow C$ by

$$\begin{aligned}A_r(u) &= \{z\in C: h(z,v)+\zeta(v)-\zeta(v)+\langle v-z,Az\rangle\\ &+ \frac{1}{r}\langle v-z,Jz-Ju\rangle\geq 0,\forall v\in C\},u\in E,\end{aligned}$$

the mapping A_r has the following properties:
(a) A_r is single-valued;
(b) A_r is a firmly nonexpansive mapping, that is
$\langle A_rx-A_ry,JA_rx-JA_ry\rangle\leq\langle A_rx-A_ry,Jx-Jy\rangle\ \forall x,y\in E$
(c) $F(A_r)=GMEP(h,A,\zeta)=\hat{F}(A_r)$
(d) $GMEP(h,A,\zeta)$ is a closed convex set of C;
(e) $\phi(q,A_ru)+\phi(A_ru,u)\leq\phi(q,u),\forall q\in\hat{F}(A_r),u\in E.$
where $\hat{F}(A_r)$ denotes the set of asymptotic fixed points of A_r, i.e., $\hat{F}(A_r):=\{x\in C:\exists\{x_n\}\subset C$, such that $x_n\rightharpoonup x,\|x_n-A_rx_n\|\longrightarrow 0.(n\longrightarrow\infty)\}$.

Lemma 11: Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let $\tau: C\times C\longrightarrow\mathbb{R}$ ba as in Lemma 9 above. Let $T_r:E\longrightarrow C$, for $r>0$ and $x\in E$, be defined by

$$T_r(x)=\{u\in C:\tau(u,y)+\frac{1}{r}\langle y-u,Ju-Jx\rangle\geq 0,\forall y\in C\},$$

Then, the following hold:
a) T_r is single-valued
b) T_r is firmly nonexpansive-type mapping i.e

$$\langle T_rx-T_ry,JT_rx-JT_ry\rangle\leq\langle T_rx-T_ry,Jx-Jy\rangle,\forall x,y\in E$$

c) $F(T_r)=GMEP(f,A,\varphi)=\hat{F}(T_r)$
d) $GMEP(f,A,\varphi)$ is closed and convex subset of C
e) $\phi(p,T_rx)+\phi(T_rx,x)\leq\phi(p,x)$

Proof: We divide the proof into several steps:
a) T_r is single-valued . Indeed, for $x\in E$ and $r>0$, let $u_1,u_2\in T_rx$

$$\tau(u_1,u_2)+\frac{1}{r}\langle u_2-u_1,Ju_1-Jx\rangle\geq 0,$$

and

$$\tau(u_2, u_1) + \frac{1}{r}\langle u_1 - u_2, Ju_2 - Jx\rangle \geq 0,$$

Adding the two inequalities, we have

$$\tau(u_1, u_2) + \tau(u_2, u_1) + \frac{1}{r}\langle u_2 - u_1, , Ju_1 - Ju_2\rangle \geq 0,$$

From (A_2) and $r > 0$, we have

$$\langle u_2 - u_1, , Ju_1 - Ju_2\rangle \geq 0,$$

Since E is strictly convex, we have

$$u_1 = u_2$$

b) T_r is a firmly nonexpansive-type mapping . Now for $x, y \in C$, we have

$$\tau(T_r x, T_r y) + \frac{1}{r}\langle T_r y - T_r x, JT_r x - Jx\rangle \geq 0,$$

and

$$\tau(T_r y, T_r x) + \frac{1}{r}\langle T_r x - T_r y, JT_r y - Jy\rangle \geq 0,$$

Adding the two inequalities, we have

$$\begin{aligned}\tau(T_r x, T_r y) + \tau(T_r y, T_r x) \quad &+ \quad \frac{1}{r}\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy\rangle \\ &\geq \quad 0,\end{aligned}$$

From (A_2) and $r > 0$, we have

$$\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy\rangle \geq 0,$$

Therefore, we have

$$\langle T_r y - T_r x, JT_r x - JT_r y\rangle \leq \langle T_r y - T_r x, Jx - Jy\rangle.$$

c) $T_r = GMEP(f, A, \varphi)$. Indeed, we have

$$\begin{aligned} z \in F(T_r) \quad &\Longleftrightarrow \quad z = T_r \\ &\Longleftrightarrow \quad (z, y) + \frac{1}{r}\langle y - z, Jz - Jz\rangle \geq 0 \\ &\Longleftrightarrow \quad \tau(z, y) \geq 0, \forall y \in C \\ &\Longleftrightarrow \quad z \in GMEP(f, A, \varphi)\end{aligned}$$

d) $GMEP(f, A, \varphi)$ is closed and convex .
Now from (c) $GMEP(f, A, \varphi) = F(T_r)$ and (b), we have

$$\langle T_r y - T_r x, JT_r x - JT_r y\rangle \leq \langle T_r y - T_r x, Jx - Jy\rangle.$$

Moreover, we have

$$\begin{aligned}\phi(T_rx,T_ry)+\phi(T_ry,T_rx) &= 2\|T_rx\|^2-2\langle T_rx,T_ry\rangle\\ &- 2\langle T_ry,JT_rx\rangle+2\|T_ry\|^2\\ &= 2\langle T_rx,JT_rx-JT_ry\rangle+2\langle T_ry,JT_ry-JT_rx\rangle\\ &= 2\langle T_rx-T_ry,JT_rx-JT_ry\rangle\end{aligned}$$

and

$$\begin{aligned}\phi(T_rx,y)+\phi(T_ry,x)-\phi(T_rx,x)-\phi(T_ry,y) &= \|T_rx\|^2\\ &- 2\langle T_rx,Jy\rangle+\|y\|^2+\|T_ry\|^2-2\langle T_ry,Jx\rangle+\|x\|^2\\ &- \|T_rx\|^2+2\langle T_rx,Jx\rangle-\|x\|^2\\ &-\|T_ry\|^2+2\langle T_ry,Jy\rangle-\|y\|^2\\ &= 2\langle T_rx,Jx-Jy\rangle\\ &-2\langle T_ry,Jx-Jy\rangle\\ &= 2\langle T_rx-T_ry,Jx-Jy\rangle.\end{aligned}$$

Hence, we have

$$\begin{aligned}\phi(T_rx,T_ry)+\phi(T_ry,T_rx) &\leq \phi(T_rx,y)+\phi(T_ry,x)\\ &- \phi(T_rx,x)-\phi(T_ry,y)\end{aligned}\qquad(2)$$

So, we have, for $x,y\in C$

$$\phi(T_rx,T_ry)+\phi(T_ry,T_rx)\leq\phi(T_rx,y)+\phi(T_ry,x)$$

Taking $y=z\in F(T_r)$, we obtain

$$\phi(z,T_rx)\leq\phi(z,x)$$

Next, we show that $\hat{F}(T_r)=GMEP(f,A,\varphi)$. Let $p\in F(T_r)$, then there exists $\{z_n\}\subset E$ such that $z_n\rightharpoonup p$ and $\lim_{n\to\infty}(z_n-T_rz_n)=0$. Moreover, we obtain $T_rz_n\rightharpoonup p$. Hence, we have $p\in C$, since J is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty}\frac{1}{r}\|Jz_n-JT_rz_n\|=0.$$

From the definition of T_r, we have

$$\tau(T_rz_n,y)+\frac{1}{r}\langle y-T_rz_n,JT_rz_n-Jz_n\rangle\geq 0.$$

Since

$$\frac{1}{r}\langle y-T_rz_n,JT_rz_n-Jz_n\rangle\geq-\tau(T_rz_n,y)\geq\tau(y,T_rz_n)$$

and τ is lower semi-continuous and convex in the second variable, we have

$$0 \geq \lim_{n\to\infty} \tau(y, T_r z_n) \geq \tau(y, p)$$

Therefore, we have

$$\tau(y, p) \leq 0, \forall y \in C.$$

Let $y \in C$ and set $x_t = ty + (1-t)p$, for $t \in (0, 1]$. Then, we have

$$\begin{aligned} 0 &= \tau(x_t, x_t) \\ &\leq t\tau(x_t, y) + (1-t)\tau(x_t, p) \\ &\leq t\tau(x_t, y) \end{aligned}$$

Dividing by t, we have

$$\tau(x_t, y) \geq 0$$

Letting $t \downarrow 0$, we have from (A_3) that

$$\tau(p, y) \geq 0, \forall y \in C$$

This implies that $p \in GMEP(f, A, \varphi)$
and hence $F(T_r) = GMEP(f, A, \varphi) = \hat{F}(T_r)$. Thus,$T_r$ is a relatively nonexpansive mapping.
$F(T_r) = GMEP(f, A, \varphi)$ is closed and convex

e) $\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x)$. From (b) and (2), for each $x, y \in E$, we obtain

$$\begin{aligned} \phi(T_r x, T_r y) + \phi(T_r y, T_r x) &\leq \phi(T_r x, y) + \phi(T_r y, x) \\ &- \phi(T_r x, x) - \phi(T_r y, y). \end{aligned}$$

Letting $y = p \in F(T_r)$, we obtain

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

3. MAIN RESULS

We introduce an iterative scheme for finding a common element of the set of solution of the generalized mixed equilibrium problems and the set of common fixed points of infinite family of multivalued asymptotically quasi nonexpansive mapping in Banach space.

Theorem 1: Let E be a uniformly smooth and uniformly convex Banach space, and Let C be a nonempty closed convex subset of E and $\hat{C}B(C)$ be the class of nonempty, closed and bounded subsets of C. Let $f : C \times C \longrightarrow \mathbb{R}$ be bi functions which satisfy the conditions $(A_1) - (A_4)$, $A : C \longrightarrow E^*$ be continuous and monotone. Let $\varphi : C \longrightarrow \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semi-continuous function. Let $\{T_i\}$ be a sequence of continuous multivalued asymptotically quasi nonexpansive mappings from C into $\hat{C}B(C)$ with $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi) \neq \emptyset$, for which $T_i x^* = \{x^*\}, \forall x^* \in \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi)$. Let $\{\alpha_{i,n}\}_{n=1}^{\infty}, i = 0, 1, 2, 3, ...$ be sequences in $(0, 1)$ such that $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$ for all $n \geq 1$. Let $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{j,n} > 0$, for all $j \in N$. Let $\{r_n\} \subset [a, \infty]$, for some $a > 0$, where T_i^n stand for the n compositions of T_i. Assume that the condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is satisfied .

Then the sequence $\{x_n\}$ generated by (1) converges strongly to an element of F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof: For $r > 0$, let functions $\tau : C \times C \longrightarrow \mathbb{R}$ and $T_r : E \longrightarrow C$ be defined by

$$\tau(x, y) = \sum_{j=1}^{K} f_j(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall x, y \in C$$

and

$$T_r(x) = \{u \in C : \tau(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \forall y \in C\}, \forall x \in E$$

respectively. Now by Lemma 10, the function τ satisfies conditions $(A1)-(A4)$ and T_r has the properties $(a)-(d)$. Therefore scheme(1) can be rewritten as

$$\begin{cases} u_1 \in E \\ x_n \in C \textit{ such that } \tau(x_n, y) + \dfrac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \ \forall y \in C \\ u_{n+1} = \alpha_{0,n} x_n + \displaystyle\sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, \ w_{i,n} \in T_i^n x_n, \ n \geq 1. \end{cases}$$

We divide the proof into a number of steps:

Step 1: We prove that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0$, for all $i \in \mathbb{N}$. Let $x^* \in F$ and notice that $x_n = T_{r_n} u_n$, then $\|x_n - x^*\| = \|T_{r_n} u_n - x^*\| \leq \|u_n - x^*\|$, we obtain the following estimations;

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|T_{r_{n+1}} u_{n+1} - x^*\| \\
&\leq \|u_{n+1} - x^*\| \\
&= \|\alpha_{0,n}(x_n - x^*) + \sum_{i=1}^{\infty} \alpha_{i,n}(w_{i,n} - x^*)\| \\
&\leq \alpha_{0,n}\|x_n - x^*\| + \sum_{i=1}^{\infty} \alpha_{i,n}\|w_{i,n} - x^*\|
\end{aligned}$$

$$\begin{aligned}
&= \alpha_{0,n}\|x_n - x^*\| + \sum_{i=1}^{\infty} \alpha_{i,n}(d(w_{i,n}, T_i^n x^*)) \\
&\leq \alpha_{0,n}\|x_n - x^*\| + \sum_{i=1}^{\infty} \alpha_{i,n}(H(T_i^n x_n, T_i^n x^*)) \\
&\leq \alpha_{0,n}\|x_n - x^*\| + \sum_{i=1}^{\infty} \alpha_{i,n}[k_n\|x_n - x^*\|] \\
&= \alpha_{0,n}\|x_n - x^*\| + k_n \sum_{i=1}^{\infty} \alpha_{i,n}\|x_n - x^*\| \\
&= (\alpha_{0,n} + k_n \sum_{i=1}^{\infty} \alpha_{i,n})\|x_n - x^*\| \\
&\leq (k_n\alpha_{0,n} + k_n \sum_{i=1}^{\infty} \alpha_{i,n})\|x_n - x^*\| \\
&= k_n(\alpha_{0,n} + \sum_{i=1}^{\infty} \alpha_{i,n})\|x_n - x^*\| \\
&= k_n\|x_n - x^*\| \\
&= (1 + (k_n - 1))\|x_n - x^*\|.
\end{aligned}$$

Hence, we have

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + (k_n - 1)\|x_n - x^*\|. \tag{3}$$

Hence, by using Lemma 7, $\sum_{i=1}^{\infty}(k_n - 1) < \infty$, we conclude that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Then $\{x_n\}$ is bounded and so are $\{w_{i,n}\}$, $\{u_n\}$ and $\{T_i^n x_n\}$.

Now by using Lemma 5 we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|T_{r_{n+1}}u_{n+1}-x^*\|^2 \\
&\leq \|u_{n+1}-x^*\|^2 \\
&= \|\alpha_{0,n}x_n+\sum_{i=1}^{\infty}\alpha_{i,n}w_{i,n}-x^*\|^2 \\
&= \|\alpha_{0,n}(x_n-x^*)+\sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x^*)\|^2 \\
&\leq \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\|w_{i,n}-x^*\|^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\| \\
&= \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}(d(w_{i,n},T_i^nx^*))^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&\leq \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}(H(T_i^nx_n,T_i^nx^*))^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&\leq \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}[k_n\|x_n-x^*\|]^2 \\
&\quad -\alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&= \alpha_{0,n}\|x_n-x^*\|^2+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n}\|x_n-x^*\|^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&= (\alpha_{0,n}+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&\leq (k_n^2\alpha_{0,n}+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&= k_n^2(\alpha_{0,n}+\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2 \\
&- \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&= k_n^2\|x_n-x^*\|^2-\alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) \\
&= (1+(k_n^2-1))\|x_n-x^*\|^2-\alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|).
\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq \|x_n-x^*\|^2 + (k_n^2-1)\|x_n-x^*\|^2 \\ - \alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|).\end{aligned} \tag{4}$$

Now from (4), we have

$$\begin{aligned}\alpha_{0,n}\alpha_{i,n}g(\|x_n-w_{i,n}\|) &\leq \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &+ (k_n^2-1)\|x_n-x^*\|^2\end{aligned}$$

By the assumption that $\liminf_{n\to\infty}\alpha_{0,n}\alpha_{i,n} > 0$, we have $\lim_{n\to\infty} g(\|x_n - w_{i,n}\|) = 0$, then by continuity of g, we have

$$\lim_{n\to\infty}\|x_n-w_{i,n}\| = 0. \tag{5}$$

Therefore

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) \leq \lim_{n\to\infty}\|x_n-w_{i,n}\| = 0.$$

This implies that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0, \tag{6}$$

for all $i \in \mathbb{N}$.

Step 2: We show that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$.

Now, from *step*1 by using (3) with $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and for each $x^* \in F$, then the $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This implies that $d(x_{n+1}, F) \leq (1+(k_n-1))d(x_n, F)$, then by lemma 7, $\lim_{n\to\infty} d(x_n, F)$ exists. Hence by the assumption $\liminf_{n\to\infty} d(x_n, F) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{7}$$

Now, let $x^* \in F$ be arbitrary, then

$$\begin{aligned}d(x_n, T_i x_n) &\leq d(x_n, T_i^n x_n) + d(T_i^n x_n, x^*) + d(x^*, T_i x_n) \\ &\leq d(x_n, T_i^n x_n) + H(T^n i x_n, T_i^n x^*) + H(Tix_n, T_i x^*) \\ &\leq d(x_n, T_i^n x_n) + k_n\|x_n-x^*\| + k_1\|x_n-x^*\| \\ &= d(x_n, T_i^n x_n) + (k_n+k_1)\|x_n-x^*\|.\end{aligned} \tag{8}$$

Therefore,

$$d(x_n, T_i x_n) \leq d(x_n, T_i^n x_n) + (k_n+k_1)\|x_n-x^*\|.$$

Taking infimum over all $x^* \in F$, we get

$$d(x_n, T_i x_n) \leq d(x_n, T_i^n x_n) + (k_n+k_1)d(x_n, F) \tag{9}$$

Applying (6) and (7) in (9) , we obtain

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0. \tag{10}$$

Step 3: We show that the sequence $\{x_n\}$ converges strongly to a point in F.
Since $\lim_{n\to\infty} d(x_n, F) = 0$. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $b^* \in F$ such that

$$\|x_n - b^*\| < \frac{\varepsilon}{2},\ \forall n \in \mathbb{N}$$

Thus for any $m \in \mathbb{N}$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - b^*\| + \|b^* - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

This implies that

$$\|x_{n+m} - x_n\| < \varepsilon,\ \forall n \in \mathbb{N}.$$

Thus the sequence $\{x_n\}$ is cauchy. Since the space is complete, $x_n \to \hat{x}$ as $n \to \infty$.
Finally, we show that $\hat{x} \in F$. Clearly for any $i \in \mathbb{N}$

$$d(\hat{x}, T_i\hat{x}) = \lim_{n\to\infty} d(x_n, T_i x_n) = 0.$$

This implies that $\hat{x} = T_i\hat{x}$, thus

$$\hat{x} \in \cap_{i=1}^{\infty} F(T_i).$$

Next, we show that $\hat{x} \in GMEP(f, A, \varphi)$. Now firstly, we show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Let $x^* \in F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi)$ and by using Lemma 8, we have

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|T_{r_n}u_n - T_{r_n}x^*\|^2 \\
&\leq \langle T_{r_n}u_n - T_{r_n}x^*, j(u_n - x^*)\rangle \\
&= \langle x_n - x^*, j(u_n - x^*)\rangle \\
&\leq \frac{1}{2}[\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - g(\|u_n - x_n\|)
\end{aligned}$$

This implies that

$$\|x_n - x^*\|^2 \leq \|u_n - x^*\|^2 - g(\|u_n - x_n\|). \tag{11}$$

Using (1) and (11), we have

$$\begin{aligned}
\|u_{n+1}-x^*\|^2 &= \|\alpha_{0,n}x_n+\sum_{i=1}^{\infty}\alpha_{i,n}w_{i,n}-x^*\|^2\\
&= \|\alpha_{0,n}(x_n-x^*)+\sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x^*)\|^2\\
&\le \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\|w_{i,n}-x^*\|^2\\
&= \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}(d(w_{i,n},T_i^nx^*))^2\\
&\le \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}(H(T_i^nx_n,T_i^nx^*))^2\\
&\le \alpha_{0,n}\|x_n-x^*\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}[k_n\|x_n-x^*\|]^2\\
&= \alpha_{0,n}\|x_n-x^*\|^2+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n}\|x_n-x^*\|^2\\
&= (\alpha_{0,n}+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2\\
&\le (k_n^2\alpha_{0,n}+k_n^2\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2
\end{aligned}$$

$$\begin{aligned}
&= k_n^2(\alpha_{0,n}+\sum_{i=1}^{\infty}\alpha_{i,n})\|x_n-x^*\|^2\\
&= k_n^2\|x_n-x^*\|^2\\
&\le k_n^2[\|u_n-x^*\|^2-g(\|u_n-x_n\|)]\\
&= k_n^2\|u_n-x^*\|^2-k_n^2g(\|u_n-x_n\|)
\end{aligned}$$

This implies that

$$\begin{aligned}
k_n^2g(\|u_n-x_n\|) &\le k_n^2\|u_n-x^*\|^2-\|u_{n+1}-x^*\|^2\\
&= (1+(k_n^2-1))\|u_n-x^*\|^2-\|u_{n+1}-x^*\|^2\\
&= (k_n^2-1)\|u_n-x^*\|^2+\|u_n-x^*\|^2-\|u_{n+1}-x^*\|^2
\end{aligned}$$

Therefore, since $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $\lim_{n\to\infty} \|u_n - x^*\|$ exists, we have

$$\lim_{n\to\infty} k_n^2 g(\|u_n - x_n\|) = 0. \qquad (12)$$

It follows from the property of g that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \qquad (13)$$

By using (13) and since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \qquad (14)$$

From the assumption $r_n \in [a, \infty)$ and $a > 0$, we have

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0.$$

Since $\{x_n\}$ is bounded and so is $\{Jx_n\}$, there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ such that $Jx_{n_k} \rightharpoonup \hat{x}$, since $\{u_n\}$ is bounded, by (14), we also obtain $Ju_{n_k} \rightharpoonup \hat{x}$. Putting $x_n = T_{r_n} u_n$, we have

$$\tau(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \forall y \in C.$$

Replacing n by n_k, and using (A_2) we have

$$\frac{1}{r_n}\langle y - x_{n_k}, Jx_{n_k} - Ju_{n_k}\rangle \geq -\tau(x_{n_k}, y) \geq \tau(y, x_{n_k}), \forall y \in C.$$

Letting $k \longrightarrow \infty$, we have from $Jx_{n_k} \rightharpoonup \hat{x}$ and (A_4) that

$$\tau(y, \hat{x}) \leq 0, \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and $\tau(y_t, \hat{x}) \leq 0, \forall y \in C$, so from (A_1) and (A_3), we have

$$\begin{aligned} 0 &= \tau(y_t, y_t) \\ &\leq t\tau(y_t, y) + (1-t)\tau(y_t, \hat{x}) \\ &\leq t\tau(y_t, y). \end{aligned}$$

Dividing by t, we have

$$\tau(y_t, y) \geq 0, \forall y \in C.$$

Letting $t \longrightarrow 0$, and using (A_3), we have

$$\tau(\hat{x}, y) \geq 0, \forall y \in C.$$

Therefore $\hat{x} \in GMEP(f, A, \varphi)$. Hence, we have $\hat{x} \in F$.

Theorem 2: Let E be a uniformly convex real Banach space and C

a nonempty, closed and convex subset of E. Let $\{T_i\}$ be a sequence of continuous multivalued asymptotically quasi-nonexpansive mappings from C into $\hat{C}B(C)$ with $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi) \neq \emptyset$, for which $T_i x^* = \{x^*\}$, $\forall x^* \in \cap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi)$. Let $\{T_i\}$ be hemi compact for at least one $i \in \mathbb{N}$ and T_i is continuous for each $i \in \mathbb{N}$, let $\{\alpha_{i,n}\}_{i=1}^{\infty}, i = 1, 2, 3, ...$ be a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \alpha_{i,n} = 1$ for all $n \geq 1$. Let $\{x_n\}$ be a sequence defined by (1) with $\liminf_{n\to\infty} \alpha_{i,0}\alpha_{j,n} > 0$ for all $j \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point in F.

Proof: From Theorem 1, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $\forall i \in \mathbb{N}$. Now, let T_{i_o} be hemi compact for some $i_o \in \mathbb{N}$, then $\lim_{n\to\infty} d(x_n, T_{i_o} x_n) = 0$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow q$ as $k \longrightarrow \infty$ for some $q \in C$. By continuity of T_i for each $i \in \mathbb{N}$, we get $\lim_{k\to\infty} d(x_{n_k}, T_i x_{n_k}) = d(q, T_i q)$. This implies that $d(q, T_i q) = 0$ and $q \in F$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. This complete the proof.

Theorem 3: Let $E, C, \{T_i\}$,$\{\alpha_{i,n}\}$ and $\{x_n\}$ be as in Theorem 1, where C is a compact convex subset of E. Then $\{x_n\}$ converges strongly to a point in F.

Proof: By Theorem 1, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \forall i \in \mathbb{N}$. From the compactness of C, there exists a subsequence $\{x_{n_k}\}_{n=k}^{\infty}$ of $\{x_n\}$ such that
$\lim_{k\to\infty} \|x_{n_k} - q\| = 0$ for some $q \in C$. Thus for $i \in \mathbb{N}$, we have

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i q) \\ &\leq \|x_{n_k} - q\| + d(x_{n_k}, T_i x_{n_k}) + k_1 \|x_{n_k} - q\| \\ &= (1 + k_1)\|x_{n_k} - q\| + d(x_{n_k}, T_i x_{n_k}). \end{aligned}$$

Then,

$$d(q, T_i q) = 0, q \in F(T_i), \ \forall i \in \mathbb{N}.$$

Hence, since $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$ and using the conclusion of Theorem 1, we have $\lim_{n\to\infty} \|x_n - q\| = 0$. i.e $x_n \longrightarrow q, q \in F$. This complete the proof.

ACKNOWLEDGEMENTS

The authors wish to thank the referees for careful reading and pointing out vital observations that lead to the improvement of the presentation of the paper.

REFERENCES

[1] M. Abbas and B.E. Rhoades, *Fixed point theorems for two new classes of multivalued mappings,* Appl. Math. Lett **22** (9) 1364 -1368. 2009.

[2] Y.I. Alber, *Metric and generalised projection operators in Banach spaces properties and applications, in; A.G, Kartosatos (Ed), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,* Marcel Dekker, New York, 15-50. 1996.

[3] A. Bunyawat and S. Suantai, *Convergence theorems for infinite family of multivalued quasi-nonexpansive mappings in uniformly convex Banach spaces,* Abstract and Applied Analysis **2012** (6) 2012. Article ID 435790.

[4] E. Blum and W. Oettli, *From Optimization and Variational inequalities to Equilibrium Problems* Mathematics Students **63** 123-145, 1994.

[5] S.S. Chang, J.K. Kim and X.R. Wang *Modified block iterative algorithm for solving convex feasibility problem in Banach spaces,* J. inequaland Application **2010** (14), 2010. Article ID869684.

[6] T.C. Deng, T. Chen and Y.L Yin, *Strong convergence theorems for mixed equilibrium problem and Asymptotically I- Nonexpansive Mapping in Banach spaces,* Abstract and Applied Analysis **2014**(12), 2014. Article ID 965737.

[7] J.N. Ezeora, *Convergence theorem for generalized mixed equilibrium problem and common fixed point problem for a family of multivalued mappings,* International Journal of Analysis and applications **10**(1), 48-57, 2016.

[8] I. Gornicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Commentationes Mathematics Universitatis Caroliner **30**(2), 249-252, 1989.

[9] J.C Huang, *A general principle for Ishikawa iterations for multivalued mappings,* Indian Journal of pure and Applied Mathematics,**2**(8) 1091-1098, 1997.

[10] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J, Optim, **13**(3) 938-945, 2002.

[11] S.B. Nadler Jr, *Multivalued contraction mapping*, Pacific Journal of mathematics, **30**(1969), 475-488, 1969.

[12] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. 73 (1967), 591-597.

[13] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces,* Computer and Mathematics with Applications, **54**(6), 872-877, 2007.

[14] K.P.R. Sastry and G.V.R. Babu, *Convergence of Ishikawa iterates for a multivalued mapping with a fixed point*, Czechoslovak Mathematics Journal **55**(4), 817-826, 2005.

[15] J. Schu, *Weak and Strong convergence to fixed points of asymptotically nonexpansive mappings*, Bulletin of the Australian Mathematical society, **43**(1), 153-159, 1991.

[16] N. Shahzad and H. Zegeye, *Mann and Ishikawa iterative scheme for multivalued maps in Banach spaces*, Nonlinear Analysis, Theory Methods and Applications, **71**(3), 838-844, 2009.

[17] Y. Song and H. Wang , *Convergence of iterative algorithms for multivalued mappings in Banach spaces*, Nonlinear Analysis, Theory Methods and Applications, **70**(4), 1547-1556, 2009.

[18] W. Takahashi and K. Zembayashi, *Weak and strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach space*,J. Nonlinear Analysis ,**70**(2009), 45-57, 2009.
[19] K. Tan and H.K. Xu, *Approximation fixed points of nonexpansive mappings by the Ishikawa iteration process,* Journal of mathematical Analysis and Application, **178**(2), 301-308, 1993.
[20] S.S. Zhang, *Generalized mixed equilibrium problem in Banach spaces,* Appl. Math. Mech, **30**(9), 1105 - 1112, 2009.

DEPARTMENT OF MATHEMATICAL SCIENCES, BAYARO UNIVERSITY, KANO, NIGERIA.
E-mail address: bashiralik@yahoo.com

DEPARTMENT OF MATHEMATICS, FEDERAL COLLEGE OF EDUCATION, ZARIA, NIGERIA.
E-mail address: lawalu4@gmail.com