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VERTEX RELATIONS OF ORDER DIVISOR GRAPHS OF SUBGROUPS OF FINITE GROUPS

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ABSTRACT. Let $S(G)$ be the set of subgroups of a finite group G. The order divisor graph of $S(G)$ denoted by $\Gamma(S(G))$ is a simple graph whose vertex set is $S(G)$ such that two vertices $H, K \in S(G)$ and $H \neq K$ are adjacent if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$. In this paper, we study the relationships between the vertices of the order divisor graphs of subgroups of finite groups. We show that if H is a subgroup of a finite group G, then the degree of vertex $H \in V(S(G))$ is greater or equal to 2. Furthermore, we also establish that there exists a path between any two arbitrary vertices of $\Gamma(S(G))$; and there is atleast $(|S(G)|-2)$ cycles of length 3 in the graph $\Gamma(S(G))$ if $\Gamma(S(G))$ is a non triangle-free order divisor graph. Our results complement and extend some results in literature.

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1. INTRODUCTION

Divisor graph was introduced in 2000 by Singh and Santhosh[1], they defined divisor graph as an ordered pair (V, E) where $V \subset \mathbb{Z}$ and for all $u, v \in V$, $u \neq v$, $uv \in E$ if and only if $u|v$ or $v|u$; they further showed that every odd cycle of length $l \geq 5$ is not a divisor graph. After then, Chartrand, Muntean, Saenpholphant and Zhang [2] further studied the divisor graphs, their major contribution was the validation of the work of [1]. Both [1] and [2] considered divisor graphs of finite sets where the vertex set corresponds with the collection of subsets. Recently, this concept has been extended to some sophisticated algebraic structures. Daoub, Shafah, and Bribesh [5] studied divisor graph with respect to finite commutative rings, they focused on investigating the degrees

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of vertices, radius, diameter, center, cycles, peripheries, clique and some graph coloring properties of the order divisor graphs of finite commutative rings; they also proved some interesting theorems relating to these concepts. To further expand the concept of divisor graph, Hafezieh [6] introduced the bipartite divisor graph, he used the combinatorial properties of the graph to discuss the structure of groups. He further used the solvable and nonsolvable concepts of the group to established that each connected component of the bipartite divisor graph is a path.

Chalapathi and Kiran-kumar [3] studied the order divisor graphs of finite groups, the core findings of their study was to disclose in detail the interplay of group theoretic properties of a finite group G with graph theoretic properties of the order divisor graph of the group G. They further studied the density, girth and diameter of the order divisor graph of finite group. In the same vein, Rehman, Baig, Imran and Khan [4] showed that the order divisor graph of a finite group is a star graph if and only if every non-identity element of G has prime order. However, for abelian group G, the order divisor graph is a star graph if and only if G is elementary abelian.

Motivated by the work of Chalapathi and Kiran-kumar [3] and other recent research in this direction, we study the relationships between the vertices of the order divisor graphs of subgroups of finite groups. We establish that if H is a subgroup of a finite group G, then the degree of vertex $H \in V(S(G))$ is greater or equal to 2. Furthermore, we also show that there exists a path between any two arbitrary vertices of $\Gamma(S(G))$; and there is atleast $(|S(G)| - 2)$ cycles of length 3 in the graph $\Gamma(S(G))$ if $\Gamma(S(G))$ is a non triangle-free order divisor graph. Our results complement and extend recent results in literature.

2. PRELIMINARY

In this section, we present in brief some basic definitions, terms and concepts in graph theory. We also state some known and useful results which will be needed in the proof of our main results and understanding of this paper. For details about these basic definitions, terms and results given in this section, we refer the readers to [7, 8, 9, 13].

A graph Γ is a combination of a nonempty set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of elements called edges to form a combinatorial structure. A graph with no loops and multiple edges is called a simple graph, the order and size of a graph are $|V(\Gamma)|$ and $|E(\Gamma)|$ respectively. For any vertex u in a graph Γ, the degree of the vertex u denoted by $deg(u)$ is the number of edges incident with the vertex u. A graph in which all vertices have the same degree is called a regular graph. A graph Γ is called connected if there is a path between any two distinct vertices in Γ. A graph Γ is complete if every two distinct vertices in Γ are adjacent. A complete graph with n vertices is denoted by K_n. It is worthy of note that throughout this article, we shall study only simple and undirected graphs. Henceforth, all graphs are simple and undirected.

Definition 2.1. [3] Let $S(G)$ be the set of the subgroups of a finite group G. The order divisor graph of $S(G)$ denoted by $\Gamma(S(G))$ is a simple graph whose vertex set is $S(G)$ such that two vertices $H, K \in S(G)$ and $H \neq K$ are adjacent if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$, where $|H|$, $|K|$ denote the order of subgroups H and K respectively.

Lemma 2.2. *(Lagrange's theorem)* [7] *Let H be a subgroup of a finite group G. Then the order of H divides the order of G.*

Definition 2.3. [8] A prime number p is an integer greater than 1 whose only positive divisors are p and 1. A positive integer which is not prime is called composite

Definition 2.4. [13] A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk.

Definition 2.5. [13] A walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.

Definition 2.6. [10] A cycle is a closed trail in a graph in which the first vertex is equal to the last vertex; the first vertex is the only vertex that is repeated.

Definition 2.7. [12] A triangle-free graph is a graph containing no graph cycles of length three.

Definition 2.8. [11] The girth of a graph Γ, denoted as $gir(\Gamma)$ is the size of the smallest cycle in the graph. The odd girth of a graph is the size of the smallest odd cycle in the graph.

Definition 2.9. [13] A Subgroup H of a group G is a nonempty subset of G that forms a group under the binary operation of G.

Theorem 2.10. [3] *Let G be a group of composite order. Then the order divisor graph of $S(G)$ of G must contain a cycle of length 3.*

Theorem 2.11. [3] *Let G be a finite group of order greater than 1, p be a prime number and $\Gamma(S(G))$ be the order divisor graph of subgroups of G. Then the girth of the graph $\Gamma(S(G)$ is given by*

$$gir(\Gamma(S(G))) = \begin{cases} \infty, & \text{if } |G| = p; \\ 3, & \text{if } |G| \neq p. \end{cases}$$

Definition 2.12. [9] A graph $\Gamma(H)$ is a subgraph of a graph $\Gamma(G)$ if $V(\Gamma(H)) \subseteq V(\Gamma(G))$ and $E(\Gamma(H)) \subseteq E(\Gamma(G))$. In this case, we write $\Gamma(H) \subseteq \Gamma(G)$ and we say $\Gamma(G)$ contains $\Gamma(H)$.

3. MAIN RESULTS

This section discusses the vertex characterization of order divisor graph of subgroups of finite groups. We begin with the following examples.

Example 3.1. Let $(\mathbb{Z}_6, +)$ be the finite group of integer modulo 6 under addition. By Definition 2.1, $V(\Gamma(S(\mathbb{Z}_6))) = \{\mathbb{Z}_6, \mathbb{Z}_3, \mathbb{Z}_2, e\}$ and $E(\Gamma(S((\mathbb{Z}_6))) = \{(\mathbb{Z}_6, e), (\mathbb{Z}_6, \mathbb{Z}_2), (\mathbb{Z}_6, \mathbb{Z}_3), (\mathbb{Z}_3, e), (\mathbb{Z}_2, e)\}$ are the vertex set and edge set of $\Gamma(S(\mathbb{Z}_6))$ respectively. Then the order divisor graph $\Gamma(S(\mathbb{Z}_6))$ is as follows:

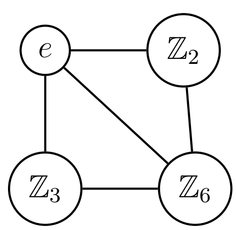

Figure 1: Order divisor graph of $S(\mathbb{Z}_6)$

Theorem 3.2. *Let G be a finite group and $S(G)$ be the set of subgroups of G. Let $\Gamma(S(G))$ be the order divisor graph of $S(G)$, then the edge set $E(\Gamma(S(G)))$ is never empty.*

Proof: Let G be a finite group with $S(G)$ as the set of the subgroups of G. Let $\Gamma(S(G))$ be the order divisor graph of $S(G)$. To show that $E(\Gamma(S(G))) \neq \emptyset$, it suffices, if we established that there exists a pair of subgroups H, K of G such that $H \neq K$ which are adjacent. From Definition 2.1, it's obvious that the trivial subgroups of G (e and G itself) satisfies the definition of order divisor graph, thus the proof. $\square$

Remark 3.3. The order divisor graph of subgroups of a finite group is never an empty graph.

Theorem 3.4. *Let $\Gamma(S(G))$ be the order divisor graph of subgroups of a finite group G. Then the group G itself is adjacent to other vertices $H \in V(\Gamma(S(G)))$.*

Proof: Suppose G is a finite group and $\Gamma(S(G))$ is the order divisor graph of subgroups of G where $S(G) = \{H : H$ is a subgroup of $G\}$ $= \mathrm{V}(\Gamma(S(G))$. By Lemma 2.2, the order of each $H \in S(G)$ divides the order of G, hence the proof. □

Theorem 3.5. *Let H,K be subgroups of a finite group G and let $\Gamma(S(G))$ be the order divisor graph of subgroups a finite group G. Then there exist a path between H and K.*

Proof: Let G be a finite group and let $S(G)$ be the set of all subgroups of G. The order divisor graph of $S(G)$ is a connected graph since the order of any arbitrary subgroup H in $S(G)$ is divided by the order of $e \in S(G)$. This implies the e is adjacent to every $H \in V(\Gamma(S(G)))$. Hence, there exists a path between every arbitrary pair $(H, K) \in V(\Gamma(S(G)))$. □

Theorem 3.6. *Let G be a finite group of order n, and $H, K \in S(G) = V(\Gamma(G))$ be the vertex set of an order divisor graph of the subgroups of G. Then the vertices H and K are not adjacent if the order of both H and K are different prime numbers p and q such that $p \neq q$.*

Proof: Let $H, K \in S(G) = V(\Gamma(G))$ be two vertices of prime orders p and q such that $p \neq q$. Suppose on the contrary H and K are adjacent on the order divisor graph of $S(G)$, then by Definition 2.1, either $|H|/|K|$ or $|K|/|H|$. Note that the orders of both H and K are two different primes and by Definition 2.12 neither p nor q is a divisor of each other; hence the proof. □

Theorem 3.7. *Let G be a finite group of order n and let H be a subgroup of G. Then $deg(H)$ of vertex $H \in V(\Gamma(S(G)))$ is greater or equal to 2 i.e $deg(H) \geq 2$.*

Proof: Let G be a finite group and suppose H is a subgroup of G. Then H is either a trivial or a proper subgroup of G. If it is a trivial subgroup of G, then it is either the subgroup containing only the identity e or the group G itself. These two trivial subgroups both have degrees greater than 2 because the order of the identity subgroup of G is a divisor of all number and by Theorem

3.4, the group G itself is adjacent to all other vertices. So, clearly the degrees of vertices e and $G \in V(\Gamma(S(G)))$ are greater than or equal to 2. Also, if H is a proper subgroup, then the order of H is either prime or composite and in either case, they both atleast divides the orders of the trivial subgroups e and G. Conclusively, it is safe to say the degree of H on the order divisor graph of G is always greater than or equal to 2. □

Theorem 3.8. *Let G be a finite group of order n and $S(G)$ be the set of subgroups of G, then the order divisor graph of the subgroups of G contains atleast $(|S(G)| - 2)$ cycles of length 3.*

Proof: To show that the order divisor graph of $S(G)$ of a finite group of order n have atleast $(|S(G)| - 2)$ cycles of length 3, then it suffices if we can show that all order divisor graphs of the subgroups of a finite group contains $(|S(G)| - 2)C_3$. From Theorem 2.10, the order divisor graph of $S(G)$ must contain a cycle of length 3, the question now is how many of these cycles are in each of the graph. Suppose G is a finite group of order n, then G has atleast two trivial subgroups e the identity and G itself which belongs to the set S(G), i.e. $\{e, G \in S(G)\}$. Also, let H be an arbitrary subgroup of G other than $\{e\}$ or G. By Lemma 2.2, the order of H also divides the order e, then it safe to say, there is a cycle $e - H - G - e$. If $|S(G)|$ denote the total number of the subgroups of G, then removing the two trivial subgroups, we have $(|S(G)| - 2)$ cycles. □

Remark 3.9. Let G be a finite group of order n and $S(G)$ be the set of subgroups of G, then the order divisor graph of the subgroups of G is not a triangle-free graph.

Theorem 3.10. *Let $\Gamma(H)$ and $\Gamma(K)$ be two subgraphs of an order divisor graph $\Gamma(S(G))$ of the subgroups of a finite cyclic group G then $|H \cap K| = gcd[|H|, |K|]$.*

Proof: Assume that $l = gcd[|H|, |K|]$, then obviously l divides $|H|$ and $|K|$. Since both H and K are subgraphs of G, then l also divides $|G|$. Besides, H and K are subgraphs which shares specific number of vertices, since l divides both $|H|$ and $|K|$ then l also divides $|H \cap K|$, thus, $|H \cap K| = gcd[|H|, |K|]$. □

The proof of theorem 3.10 is not convincing also, I have been made some changes, Check well to make sure all is in order.

3.1. Isomorphisms of Order Divisor Graphs of Subgroups of Finite Groups.

This section looks at conditions for order divisor graphs of subgroups of two different finite groups to be isomorphic. According to [13], the simple graphs $\Gamma(G) = (V_1, E_1)$ and $\Gamma(G') = (V_2, E_2)$ are isomorphic if there is a one - to - one and onto function f from V_1 to V_2 with the property that a and b are adjacent in $\Gamma(G)$ if and only if $f(a)$ and $f(b)$ are adjacent in $\Gamma(G')$ for all a and b in V_1. Such that a function f is called isomorphism.

Theorem 3.11. [3] *Let G and G' be two finite groups. Then $G \cong G'$ if and only if $\Gamma(G)) \cong \Gamma(G'))$*

Theorem 3.12. *Let G and G' be two finite groups of equal orders and let $S(G)$ and $S(G')$ be two equivalent sets of subgroups of G and G' respectively, such that a subgroup $H \in S(G)$ is isomorphic to a subgroups $K \in S(G')$, then the order divisor graph of $S(G)$ is isomorphic to the order divisor graph of $S(G')$.*

Proof: Suppose G and G' are two finite groups of equal orders and let $S(G)$ and $S(G')$ be two equivalent sets of subgroups of G and G' respectively, such that a subgroup $H \in S(G)$ is isomorphic to a subgroups $K \in S(G')$, then obviously there is a bijection $\phi : H \in S(G) \to K \in S(G')$. Now, since ϕ is a bijection, then it is both one-to-one and onto. Note that this isomorphic map is between subgroups of finite groups; thus, there is a unique $H \in S(G)$ that is mapped to a unique $K \in S(G')$. Similarly, $\phi(S(G))$ is mapped to an image in $S(G')$. Considering the two graphs $\Gamma(S(G))$ and $\Gamma(S(G'))$, since each $H \in S(G)$ isomorphic to a $K \in S(G')$ and by Definition 2.1, these two sets of subgroups contains isomorphic vertices, the edge sets must satisfies the bijiction properties of isomorphism. therefore, it is safe to say conclusively that $\Gamma(S(G)) \cong \Gamma(S(G'))$ □

4. CONCLUSION

This study look at the relationships between vertices of the order divisor graphs of subgroups of finite groups. Clearly we show that if H is a subgroup of a finite group G, the degree of H in the order divisor graph of the subgroups of G is greater or equal to 2. Also, we establish that there is always a path between two arbitrary vertices of the graph. Moreover, we show that all vertices of the graph are adjacent to the group itself and the vertices of the non triangle-free order divisor graph of S(G) always form atleast ($| S(G) | -2$) cycles of 3.

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