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PENDULUM-LIKE SYSTEMS OF A CERTAIN CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS - A REVIEW*

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ABSTRACT. From the ideal pendulum equation of second order non-linear differential equation

$$\ddot{x} + b \sin x = 0$$

to the damped and forced equations, we look at the generalized mathematical pendulum and systems associated with it.
The works of M.L.Cartwright & J.E. Littlewood, J.O.C. Ezeilo and H.O. Tejumola are indicated. The generalization to systems with multiple-equilibria, canonical forms, reduction methods and generalizations to higher-order differential equations are highlighted.
Pendulum-like systems of third order are given as examples, for dichotomy and gradient-like solutions.
Challenges to Nigerian Mathematical Society are highly emphasised in conclusion.

Keywords and phrases: Pendulum-like systems,nonlinear differential equations, multiple-equilibria, reduction methods, dichotomy and gradient-like solutions
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Dedicated to the memory of Professor J.O.C. Ezeilo (1930-2013)

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1. Introduction

From the physical point of view, the oscillations of simple pendulum are represented by

$$\ddot{x} + \frac{g}{l} \sin x = 0 \tag{1.1}$$

where $\cdot = \frac{d}{dt}$.

On introducing a generalized form with damping and forcing terms, we have

$$\ddot{x} + a\dot{x} + \frac{g}{l} \sin x = q(t). \tag{1.2}$$

A wide class of system with infinite equilibria set belongs to the pendulum-like systems. The simplest of this class is the equation of the mathematical pendulum

$$\ddot{x} + a\dot{x} + c \sin x = \gamma, \ (a > 0, \ c > 0, \ \gamma \in (0, 1)). \tag{1.3}$$

or it's simple generalization

$$\ddot{x} + a\dot{x} + \varphi(x) = 0, \ (a > 0). \tag{1.4}$$

A generalized mathematical study of these equations is what we now have in the literature as

$$\ddot{x} + f(x)\dot{x} + g(x) = q(t, x, \dot{x}), \ (\textit{Lienard equation}), \tag{1.5}$$

and

$$\ddot{x} + F(\dot{x}) + g(x) = q(t, x, \dot{x}), \ (\textit{Rayleigh equation}), \tag{1.6}$$

and a particular revolutionary type of Lienard equation in the forms

$$\begin{aligned} \ddot{x} - k(1 - x^2)\dot{x} + x &= 0, \\ \ddot{x} - k(1 - x^2)\dot{x} + x &= bk \cos \lambda t, \end{aligned}$$

which are homogeneous and non-homogeneous *Van der Pol's equation* respectively, well studied by the British Mathematicians -

Mary Lucy Cartwright and J.E.Littlewood among others, (cf. [11]). What is really peculiar with these equations was that the investigators Cartwright and Littlewood brought a revolutionary change of direction to the use and ideas of these equations.

2. The war works of M.L. Cartwright and J.E. Littlewood

In January 1938, with the threat of war hanging over Europe, the British Government's department of Scientific and Industrial Research sent a memorandum to London Mathematical Society, appealing to Pure Mathematicians to help solve a problem involving a *tricky type of equation*. Although this was not stated in the memo, it related to top-secret developments in Radio Detection and Ranging - what was soon to become known as *Radar*.

Engineers working on the project were having difficulty with the erratic behavior of high- frequency radio waves. The need had arisen, the memo said, for "*a more complete understanding of the actual behaviour of certain assemblages of electrical apparatus*".

Could any of the London Mathematical Society's members help?

The whole development of radar in World War II depended on the high power amplifiers and it was a matter of life and death to have amplifiers that did what they were supposed to do. The soldiers were plagued with amplifiers that misbehaved. The blame was on the manufacturers for the erratic behaviour. Cartwright and Littlewood discovered that the manufacturers were not to be blamed. The equation was to blame.

Precisely, Cartwright and Littlewood worked on the nonlinear differential equations involved in the technique of radio engineering of the form:

$$\ddot{x} - k(1-x^2)\dot{x} + x = 0, \tag{2.1}$$

$$\ddot{x} - k(1-x^2)\dot{x} + x = bk\cos\lambda t, \tag{2.2}$$

which arose in connection with thermionic valves. They succeeded in showing that when k is large, all solutions of (2.1) except the trivial solution $x = 0$ converge to a periodic solution whose amplitude tends to 2 as $k \to \infty$. (cf: [10],[15]).

We also note that Cartwright and Littlewood were among the first researchers to combine topological and analytical methods to tackle differential equations and discovered many of the phenomena that later became known as "CHAOS". They proved that "chaos" could arise even in equations originating in real engineering problems, (cf:[14]).

In the words of Littlewood ([14], p.964) *In the first part of the war, Miss Cartwright and I got drawn into Van der Pol's equation ... [W]e went on and on ... with no earthly prospect of "results". Suddenly the entire vista of the dramatic fine structure of solution stared at us in the face.*

A fast note, is to see the linear second order differential equation

$$\ddot{x} + 2\beta\dot{x} + n^2x = 0, \tag{2.3}$$

which has a solution of the form

$$x(t) = ae^{-\beta t}\sin(\mu t + \alpha), \ where\ \mu^2 = n^2 - \beta^2, \ (n > \beta). \tag{2.4}$$

This gives an oscillation that dies down if $\beta > 0$, and one that increases indefinitely in amplitude if $\beta < 0$.

This then predicts the findings of Cartwright and Littlewood for

$$\ddot{x} - k(1 - x^2)\dot{x} + x = 0,$$

for a damping that changes sign.

If k is small, the solution over a limited time is approximately of the form

$$x(t) = a\sin t,$$

(while a may vary slowly). Compared with the linear case, a increases when x is small and decreases when a is large. Between these two cases, it can be shown that there is one in which a is more or less stationary, corresponding to a strictly periodic solution. Finding the stationary values of a remains a problem.

3. Entrance of J.O.C. Ezeilo and others to Higher Order differential equations, with Lyapunov Functions

Most generalization of the second order differential equations / systems was the extensions to higher order non-linear differential equations. From Cartwright's work (with Littlewood) was the contribution of her student of 1959, J.O.C. Ezeilo, who concentrated on third order and higher order non-linear differential equations. A major effort was to rewrite these equations into first order systems of equations. Methods used included the Lyapunov functions and fixed point theories.

Ezeilo's works dealt with the study of qualitative properties of solutions, which included boundedness, periodicity, almost periodicity, ultimate boundedness, convergence and instability of solutions.His main theory depended on the construction of Lyapunov functions and fixed points theories. In fact most of these involved single equilibrium points. His type's of equations included, say,

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$$

where $a,\ b$ are constants, and $h(x)$ have similar behaviour as those imposed by Cartwright on the second order equations she studied, say for example $h(0) = 0$, $\frac{h(x)}{x} > 0$ for $|x|$ large enough. In addition, the use of the Routh-Hurwitz conditions (including the generalized Routh-Hurwitz conditions) were imposed.

This same line was followed by his student H.O. Tejumola, generalizing and opening areas in higher ordered equations and systems of equations(cf. [26, 27]).

We note that the direct control methods were used in these earlier discussions, including higher order differential equations. These involved the re-writing of the equations as a single system, say

$$\dot{X} = AX - B\varphi(\sigma) + P(t, X),\ \sigma(t) = C^* X$$

and the analysis carried out, either by Lyapunov functions methods or fixed point methods.

4. The Main Questions

Despite the many breakthroughs of these eminent researchers (amongst others), it was only in 1966 that J.O.C. Ezeilo asked the question that involved multiple equilibria for third order differential equations. A generalization of the problem to higher order differential equations was still lying idle to be asked and answered with multiple equilibria. Of course, these types of equations occur in physical, engineering and control problems.

While most of the work of J.O.C. Ezeilo and his collaborators concentrated on systems with single equilibrium equations, in 1966, he however asked the basic question:

"When will the real differential equation

$$\dddot{x} + a\ddot{x} + \dot{x} + \sin x = 0 \tag{4.1}$$

in which a is a constant, have a non-trivial periodic solution for arbitrary values of a ?"(cf: [13]*).*

In 1967, Barbashin considered a more general equation of the form

$$\dddot{x} + a\ddot{x} + b\dot{x} + \varphi(x) = 0 \tag{4.2}$$

where a, b are positive constants and $\varphi(x)$ is a $2\pi-$periodic C^1 function, having zeros in $[0, 2\pi)$, and sought for the existence of non-trivial solutions.(cf:[9]).

In 2006, Afuwape [4], modified these questions and then initiated a solution as:

When will a general third-order differential equation of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + \varphi(x) = 0 \tag{4.3}$$

where a is a constant, $g(\dot{x})$ is a continuous bounded function, and $\varphi(x)$ is a $2\pi-$periodic C^1 function, having zeros 0, x_o in $[0, 2\pi)$, and at any point $x \in [0, 2\pi)$, we have $\varphi^2(x) + [\varphi'(x)]^2 \neq 0$, have non-trivial periodic solutions?

A similar problem was also studied in 2007 in a joint work by Afuwape and Castellanos, [6] for equations of the form:

$$\dddot{x} + a\ddot{x} + g_1(x)\dot{x} + \varphi(x) = 0 \tag{4.4}$$

using a non-local reduction method.

These problems gave new directions for investigating equations with multiple equilibrium points. These we shall consider now.

5. Multiple equilibria equations

The simplest differential equation which gives rise to multiple equilibria is the pendulum-like equation of the second order

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = \gamma \tag{5.1}$$

where α and γ are some positive constants.

This equation(5.1) has exhaustively been investigated. In particular the equivalent system to (5.1) is:

$$\begin{cases} \dfrac{d\theta}{dt} &= \eta \\ \dfrac{d\eta}{dt} &= -\alpha\eta - \sin\theta + \gamma \end{cases} \tag{5.2}$$

We note that the fixed points (θ_{eq}, η_{eq}) for (5.2) satisfy

$$\eta_{eq} = 0, \ \textit{and} \ \sin\theta_{eq} - \gamma = 0.$$

Thus the equilibrium set is

$$\mathcal{E} = \{(\theta_{eq}, \eta_{eq}) | \eta_{eq} = 0, \ \sin\theta_{eq} - \gamma = 0\}. \tag{5.3}$$

On this set, we have different qualitative properties and portraits for (5.2).

A very important result for different values of γ is as follows:

Theorem 1. [17] *The following are true:*

(i) If $\gamma > 1$, *for any solution of (5.2), there exist positive numbers* τ *and* ε *such that*

$$\eta(t) = \frac{d\theta}{dt} \geq \varepsilon > 0, \ for\ all\ t \geq \tau; \tag{5.4}$$

(ii) If $1 > \gamma > \gamma_{cr}(\alpha)$, *system (5.2) has equilibria stable "in the small" and solutions satisfy (5.4);*

(iii) If $0 \leq \gamma \leq \gamma_{cr}(\alpha)$, *then any solution of (5.2) tends to some equilibrium point as* $t \to \infty$.

We remark that $\gamma_{cr}(\alpha)$ is called the *critical or bifurcation value* of the parameter γ.

So much work have concentrated on calculating $\gamma_{cr}(\alpha)$. Tricomi ([21]) started the estimation of $\gamma_{cr}(\alpha)$.

Definition 1. *The solutions of (5.2) with property (5.4) are called* ***circular****.*

The most desirable situation is property (iii), in which the phase difference $\theta(t)$ always has a finite limit as $t \to \infty$, whereas the frequency difference $\dot{\theta}(t)$ vanishes as $t \to \infty$.

We also note that in the circular solutions of (5.2), we can distinguish between two types - namely

(1) the case for which there exists an integer $j \neq 0$ and a number $\tau > 0$ such that

$$\theta(0) = \theta(\tau) + 2j\pi, \ \dot{\theta}(0) = \dot{\theta}(\tau) \tag{5.5}$$

called **the cycle of the second kind**; and

(2) the non-trivial periodic solution of (5.2), called **the cycle of the first kind**.

In practice, and for greater research, the differential equations considered are of order 2 and higher orders. This leads to the direct control systems and indirect control systems.

If we change the nonlinear function $\sin\theta - \gamma$ in (5.1) to arbitrary function $\varphi(\sigma)$ and θ to σ, we shall be considering the differential equation

$$\ddot{\sigma} + \alpha\dot{\sigma} + \varphi(\sigma) = 0 \tag{5.6}$$

and the equivalent system

$$\begin{cases} \dfrac{d\sigma}{dt} & = & \eta \\ \dfrac{d\eta}{dt} & = & -\alpha\eta - \varphi(\sigma) \end{cases} \tag{5.7}$$

6. Nonlocal Reduction Method

A natural generalization of equation of (5.6) and hence of system (5.7), is the multidimensional pendulum-like system

$$\begin{cases} \dot{z} & = & Az + B\varphi(\sigma) \\ \dot{\sigma} & = & Cz + D\varphi(\sigma) \end{cases} \tag{6.1}$$

where $A \in I\!\!R^{n\times n},\ B \in I\!\!R^{n\times m},\ C \in I\!\!R^{m\times n},\ D \in I\!\!R^{m\times m}, \varphi(\sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2), \cdots, \varphi_m(\sigma_m))^T$.

Viewing $\dot{\sigma}$ as the output of the system, then we have the transfer matrix from $\varphi(\sigma)$ to $\dot{\sigma}$ as

$$K(s) = C(sI - A)^{-1}B + D.$$

Generally, the following assumptions are made:

(AS1) Matrix A has no imaginary eigenvalues, (A, B) is controllable, (A, C) is observable, and $K(0)$ is nonsingular.

(AS2) The functions $\varphi_i : I\!\!R \to I\!\!R$ is Δ_i- periodic, locally Lipschitz continuous and possesses a finite number of zeros on $[0, \Delta_i),\ i = 1, 2, \cdots, m$.

(AS3) Suppose φ_i is continuously differentiable and

$$\theta_i \le \frac{d\varphi_i(\tau)}{d\tau} \le \lambda_i,\ -\infty < \theta_i, \lambda_i < +\infty,\ i = 1, 2, \cdots, m. \tag{6.2}$$

We shall set $\Theta = diag(\theta_1, \theta_2, \cdots, \theta_m),\ \Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m)$.

The equilibrium set of (6.1) is

$$\mathcal{E} = \{(z_{eq}, \sigma_{eq}) | z_{eq} = 0,\ \varphi(\sigma_{eq}) = 0\}.$$

While the theory of stability of automatic control systems continue to develope, the system (6.1) proves to be out of its framework. (A case of using Lyapunov functions, namely, "a quadratic form" and "a quadratic form plus an integral of the nonlinearity", which is exploited in control theory are of no use for systems with periodic nonlinear functions).

One of the method used to tackle systems of the form (6.1) is the nonlocal reduction method. It is based on comparison principle. It's main idea is as follows: together with the given multidimensional system, we consider a specific system of the same type, but of a lower order (of second order, as a rule) and whose properties already have already been investigated. This latter system is called a reduction system. The information on the system is used while investigating the properties of the solutions of the given system. The nonlocal reduction method proved to be fruitful for the stability investigation of (6.1).

7. Dichotomy and Gradient-like behaviour of solutions

A general class of systems are those with infinite equilibria set. These are called the pendulum-like systems ([17]).

In this section, we introduce the dichotomy of pendulum-like differential equation. Consider the system of equations of the form

$$\dot{x} = f(t, x) \tag{7.1}$$

where $f : I\!R^+ \times I\!R^n \to I\!R^n$ is continuous and locally Lipschitz in x.

Let $\Gamma = \left\{ \sum_{j=1}^{m} k_j d_j | k_j \in Z, 1 \leq j \leq m, d_j \in I\!R^n \right\}$ with d_j linearly independent $(m \leq n)$.

Definition 2. [17] *System (7.1) is **pendulum-like** with respect to Γ if for any solution $x(t, t_0, x_0)$ of (7.1), we have*

$$x(t, t_0, x_0 + d) = x(t, t_0, x_0) + d$$

for all $t \geq t_0$ and all $d \in \Gamma$.

An interesting result for identifying a pendulum-like system is

Theorem 2. [17] *System (7.1) is pendulum-like with respect to Γ if and only if*

$$f(t, x + d) = f(t, x), \text{ for all } t \geq 0, x \in I\!R^n, \text{ and } d \in \Gamma.$$

Remark 1. *We note that if $x(t) = x_{eq} = constant$ is a solution of (7.1) that is pendulum-like, then $x(t) = x_{eq} + d$ $(d \in \Gamma)$ is a solution of (7.1). So the equilibrium set is either empty or infinite.*

Now let us consider equation (7.1) in the form of a multi-input and multi-output system of the form

$$\begin{cases} \dfrac{dz}{dt} = Az + B\varphi(\sigma) \\ \dfrac{d\sigma}{dt} = Cz + D\varphi(\sigma) \end{cases} \tag{7.2}$$

where $A \in I\!R^{n\times n}, B \in I\!R^{n\times m}, \varphi(y) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2), \cdots, \varphi_m(\sigma_m))^T$, $C \in I\!R^{m\times n}, D \in I\!R^{m\times m}$.

The transfer matrix function from the input $\varphi(\sigma)$ to the output $\dot{\sigma}$ is set as

$$K(s) = C(sI - A)^{-1}B + D.$$

The basic assumptions are those given in §6, which include $K(0) \neq 0$.

Clearly, if (x_{eq}, y_{eq}) is any equilibrium of (7.2), then the equilibrium set for (7.2) is

$$\mathcal{E} = \{(z_{eq}, \sigma_{eq}) | z_{eq} = 0, \varphi(\sigma_{eq}) = 0\}. \tag{7.3}$$

Most general Lur'e problem assumes that the nonlinearities satisfy the sector conditions,

$$\mu_{1i} \leq \varphi_i \leq \mu_{2i}.$$

It is very possible to assume that the nonlinearities are continuously differentiable and satisfy in place of the sector condition, conditions on the derivatives

$$\theta_i \leq \frac{d\varphi_i}{dt} \leq \lambda_i, \tag{7.4}$$

with $-\infty < \theta_i, \lambda_i < +\infty, (i = 1, 2, \cdots, m)$.

Definition 3. *The system (7.1) is said to be* ***dichotomous*** *if every bounded solution is convergent to a certain equilibrium point.*

Setting $\Theta = diag(\theta_1, \theta_2, \cdots, \theta_m)$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m)$, the generalized Kalman-Yacubovich-Popov (KYP) lemma ([17]), can be stated as:

Theorem 3. *If assumptions AS1 and AS2 hold, then system(7.2) is dichotomous for all matrices φ satisfying condition (7.4), if there exists diagonal matrices P and R with $R \geq 0$, and a scalar $\epsilon > 0$ such that the following frequency-domain inequality holds*

$$\begin{aligned} Re\{PK(i\omega) \quad &- \quad (i\omega I - \Theta K(i\omega))^H R(i\omega I - \Lambda K(i\omega))\} \\ &+\epsilon K^H(i\omega)K(i\omega) \leq 0 \end{aligned} \tag{7.5}$$

for all $\omega \in I\!R$.

Remark 2. *If $R \equiv 0$, inequality (7.5) becomes*

$$Re\{PK(i\omega)\} + \epsilon K^H(i\omega)K(i\omega) \leq 0 \tag{7.6}$$

for all $\omega \in I\!R$.

[2] We note that $Re\{Y\} = \frac{1}{2}(Y + Y^H)$ where Y is a complex square matrix, and Y^H is its complex conjugate transpose.

Definition 4. *The system (7.1) is said to be* ***gradient-like*** *if every solution is convergent to a certain equilibrium point.*

Theorem 4. *If assumptions AS1 and AS2 hold, then system(7.2) is gradient-like if matrix A is stable and if there exists diagonal matrices P, Q, E and R with $R \geq 0$, $E > 0$ and $Q > 0$ such that the following frequency-domain inequality holds*

$$\begin{aligned} Re\{PK(i\omega) \quad - \quad & (i\omega I - \Theta K(i\omega))^H R(i\omega I - \Lambda K(i\omega))\} \\ & + K^H(i\omega)QK(i\omega) + E \leq 0 \end{aligned} \tag{7.7}$$

for all $\omega \in I\!R$.

Remark 3. *If $R \equiv 0$, inequality (7.7) becomes*

$$Re\{PK(i\omega)\} + K^H(i\omega)QK(i\omega) + E \leq 0 \tag{7.8}$$

for all $\omega \in I\!R$ for gradient-like systems of form (7.2).

Remark 4. *For dichotomy, we do not need matrix A to be stable, while for gradient-like behaviour (i.e. global convergence), we would need that matrix A in (7.1) to be stable.*

8. Application to a third-order non-linear differential equation

Let us consider the equation

$$\dddot{x} + \alpha\ddot{x} + g(x)\dot{x} + \varphi(x) = 0 \tag{8.1}$$

where $\alpha > 0$, $g(x)$ is a continuously bounded function in $\mathbb{R}$, $\varphi(x)$ a continuously differentiable periodic odd function with period 2π in $\mathbb{R}$. Moreover, we shall assume that if $\mathbb{R}$ is segmented as a union of Π_k where

$$\Pi_k \quad = \quad \{x \in [2k\pi, 2(k+1)\pi) | k \in \boldsymbol{Z}\}, \tag{8.2}$$

$\varphi(x)$ has two zeros x_{1k}, x_{2k}, $(x_{1k} < x_{2k})$, in each segment $[2k\pi, 2(k+1)\pi)$ with $\varphi'(x_{1k}) > 0$ and $\varphi'(x_{2k}) < 0$ and such that

$$\varphi^2(x) + [\varphi'(x)]^2 \neq 0;$$

for any $x \in \Pi_k$.

Theorem 5. *Suppose that*

(i) there exist positive numbers β and μ such that

$$\beta \quad \leq \quad \frac{G(x)}{x} \leq \beta + \mu, \ (x \neq 0),$$

with $\alpha^2 > 4\beta$ and some constant parameter λ_1; satisfying $\mu < \lambda_1(\lambda_1^2 - \alpha\lambda_1 + \beta)$;

(ii) $\varphi(x)$ *satisfies assumptions* $(AS1)$ *and* $(AS2)$. *Suppose also that for some* $\lambda_2 > 0$, *we have*

$$\alpha - \sqrt{(\alpha^2 - 4\beta)} < 2\lambda_2 \quad < \quad \alpha + \sqrt{(\alpha^2 - 4\beta)},$$

and $\varphi'(\sigma_0) > \alpha\beta$, *then equation*

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + \varphi(x) = 0$$

is gradient-like, if $\alpha \geq 1$, *and* $\varphi'(\bar{x}) \neq 0$ *at every zero* $\bar{x}$ *of* $\varphi(x)$, *and* $|\varphi'(\bar{x})| < \mu$ *for some* $\mu > 0$, *for all* $x \in I\!R$.

Proof of Theorem 5:

Let $G(x) = \int_0^x g(s)ds = \beta x + \hat{g}(x)$.

In order to write (8.1) in the equivalent system form of (7.2),

$$\begin{cases} \dfrac{dz}{dt} = Az + B\varphi(\sigma) \\ \dfrac{d\sigma}{dt} = Cz + D\varphi(\sigma) \end{cases} \tag{8.3}$$

we set

$A = \begin{pmatrix} -\alpha & -\beta \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $D \equiv 0$ and $\varphi(\sigma) = [\hat{g}(z_1) + \varphi(z_1)]$.

Following [4] the transfer function is

$$K(i\omega) = \frac{\beta[(\beta - \omega^2) - i\omega]}{[(\beta - \omega^2)^2 + \omega^2\alpha^2]}.$$

With the choice of $P = -\mu, \Theta = 0, \Lambda = \mu, R = \delta > 0$, and $Q = \varepsilon > 0$, the frequency domain inequality (7.7) becomes valid with $0 < \varepsilon < \mu$. This completes the proof.

In ([2]), the system

$$\dot{X} = AX - B\varphi(\sigma) + P(t, X), \ \sigma(t) = C^* X$$

which is a form of the direct control problem form was used. Interestingly, with non-singular transformations, this can be transformed into the form of (8.3). (cf:[25]).

9. Example

We can consider the following example in system form of (7.2) ([12]) with:

$$A = \begin{pmatrix} -0.4 & 3 \\ -1 & -0.5 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ -1.4 & 1 \end{pmatrix}; C = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix};$$

$$D = \begin{pmatrix} \alpha & 1.2 \\ -2 & 1 \end{pmatrix};$$

and with $\varphi_1(y_1) = \sin(y_1) - 0.2, \varphi_2(y_2) = \sin(2y_2) - 0.1$.

Carrying out the calculations for the frequency domain inequality (7.5), it clearly shows that the system is dichotomous when $\alpha \geq 1.9$.
Remarks: We finally note that some form of equation (8.1) was discussed for dissipativity in [1], [3] and for periodic solutions in [2], but the nonlocal reduction method was not used there. Rather, the generalized Kalman-Yacubovich-Popov criteria of the frequency domain techniques were used. See for example [17]. Here, we have concentrated on the nonlocal reduction method which gives us the opportunity of reducing high order systems to the analysis of second order systems.

10. Any lessons from the works above? - Challenges to Nigerian Mathematical Society

We shall only enumerate possible challenges to Nigerian Mathematical Society:-

(1) Our Society has gone very far, not to be involve in the development of every aspect of her development. For example, from the demand in 1938 for a mathematician by the government of Britain "appealing to Pure Mathematicians to help solve a problem involving a *tricky type of equation*" so as to know fully well the problems confronting the engineers, "What are we all doing today?" Allowing engineers, social scientist, Agriculturalists, Bankers to keep APPROXIMATING solutions to problems, while we can be part of it all to discuss the ACTUAL solutions, even if we cannot explicitly write them out, but FIND where the "chaos" is taking place.

(2) Our situation in the country needs our full participation in joining the bandwagon, to diffuse the problems.

    I know that there was a big attempt by late Prof. C.O.A. Sowunmi of University of Ibadan, Ibadan in early 2006, to have a model (in it's infancy) which he labeled "Model of Corruption in Nigeria". In fact, I believe if a mathematician can get such a model, he should be able to help this nation better in terms of our set-back problems.

(3) We cry daily of the poor situations in our Educational Institutions, begging those who do not know, WHY EDUCATION SHOULD be for all, and in particular, WHY should

my son/ daughter study Mathematics to help solve a problem. There is no way the problems can be solved unless we as a Society, determine to have them solved. For more than 10 years, our well planned National Mathematical Centre, Abuja was been threatened to be closed unless we start teaching "Primary and Secondary School student to have better results in NECO, WAEC and JAMB. In between, our research and promotion of the centre as centre of excellence, tailored in a format of ICTP, Trieste went down the drains. We lost even efforts to have it put on the initial path that the founding fathers had set it on. Though we now have a possible new face in the centre, it will surely take more years to be back on the earlier path of progress, unless we can find where we ALL went wrong as the critical "bifurcation" point before the "chaos".

Ladies and Gentlemen, let us pull the Nigerian Mathematical Society to have major contributions to our society, educationally, socially, economically and Industrially. Thank you all.

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