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AN APPROXIMATE SOLUTION OF θ-BASED RICHARDS' EQUATION BY COMBINATION OF NEW INTEGRAL TRANSFORM AND HOMOTOPY PERTURBATION METHOD

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ABSTRACT. In this paper, we apply the mixture of new integral transform and homotopy perturbation method (NITHPM) to solve Richards' equation. This method is a mixture of new integral transform method and homotopy perturbation method. The nonlinear term can be easily handled by homotopy perturbation method. Some cases of the Richards' equation are solved as examples to illustrate ability and reliability of mixture of new integral transform and homotopy perturbation method. The results reveal that this method is quite capable, practically will appropriate for use in such problems and can be applied to other nonlinear problems. This method is seen as a better alternative method to some existing techniques for such realistic problems.

Keywords and phrases: Richards' equation, New Integral Transform, Homotopy Perturbation Method, He's Polynomial
2010 Mathematical Subject Classification: A80

1. INTRODUCTION

When water on the ground surface enters the soil process of infiltration occurs. The water table is bounded between the saturated and unsaturated flows where atmospheric pressure prevail. Saturated flow occurs below the water table, while unsaturated flow occurs above the water table [1]. The continuity equation is together with Darcy's law as a momentum equation, and the Richards' equation is obtained [2]. Numerical and various common, even various, analytical methods have been used to solve the Richards' equation. For examples, the Exp-Function Method [3], the Homotopy Analysis Method (HAM) [4], the Adomian Decomposition Method (ADM)

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[5, 6, 7]. There still seems to be a need for the development of analytical methods to study the problem of unsaturated multi phase flow in porous media.
A new integral transform [8] was first anticipated by Artion Kashuri and Akli Fundo to solve various differential equations. Some integral transform methods like Laplace transform [9, 10] and Sumudu transform [11, 12, 13] methods, are used to solve general nonlinear non-homogenous partial differential equations with initial conditions and use richness of these integral transform lies in their capability to transform differential equations into algebraic equations which allows easy and straightforward solution procedures.

Definition 1. *Over the set of functions*

$$F = \{f(t) \backslash \exists M, k_1, k_2 > 0, \text{ such that } |f(t)| \leq M e^{\frac{|t|}{k_i^2}}, \text{ if } t \in (-1)^i \times [0, \infty)\}, \quad (1)$$

where the constant M must be finite number, k_1, k_2 may be finite or infinite.
The new integral transform is defined by:

$$A(v) = K[u(t)] = \frac{1}{v} \int_0^\infty e^{-\frac{t}{v^2}} u(t) dt. \quad (2)$$

For more details and properties of this transform, see [8, 14, 15, 16].

In the current investigation, an analytical method known as the combination of new integral transform and homotopy perturbation method (NITHPM) has been employed to solve the problem of one-dimensional infiltration of water in unsaturated soil governed by Richards' equation. In the Section **2**, mathematical formulation of Richards' equation discussed. Basic idea of the mixture of new integral transform and homotopy perturbation method is given in Section **3**. Illustrative examples are also given in order to exhibit the effectiveness of the NITHPM for solving Richards' equation in Section **4**.

2. FORMULATION OF θ-BASED RICHARDS' EQUATION

The fundamental theories describing fluid flow through porous media were first proposed by Buckingham [17] who realized that water flow in unsaturated soil is highly dependent on water content. Buckingham introduced the concept of "conductivity", relay on water content, which is today known as unsaturated hydraulic conductivity. This equation is generally referred to as Buckingham law. Buckingham also went on to define moisture diffusivity which is the product of the unsaturated hydraulic conductivity and the slope of the soil-water characteristic curve. Almost two decades later, Richards' [2] applied the continuity equation to Buckingham's law, which itself is an expansion of Darcy's law and obtained a general partial differential equation demonstrating water flow in unsaturated, non-swelling soils with the matric potential as the single dependent variable [18]. There are usually three main forms of Richards' equation present in the literature namely the mixed formulation, the h-based formulation and the θ-based formulation, where h is the weight-based pressure potential and θ is the volumetric water content. Since Richards' equation is a general combination of Darcy's law and the continuity equation as earlier mentioned, the two relations must first be written in order to derive Richards' equation. Herein, one-dimensional infiltration of water in vertical direction of unsaturated soil is considered, for which Darcy's law and the continuity equation are given by Eqs. (3) and (4) respectively [19]:

$$q = -K\frac{\partial H}{\partial z} = -K\frac{\partial (h+z)}{\partial z} = -K\left(\frac{\partial h}{\partial z} + 1\right), \tag{3}$$

and

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z}, \tag{4}$$

where K is hydraulic conductivity, H is head equivalent of hydraulic potential, q is flux density and t is time.

The mixed form of Richards' equation is obtained by substituting Eq. (3) in Eq. (4)

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K\left(\frac{\partial h}{\partial z} + 1\right)\right]. \tag{5}$$

Eq. (5) has two independent variables: the soil water content θ, and pore water pressure head h. Obtaining solutions to this equation therefore requires constitutive relations to describe the interdependence among pressure, saturation and hydraulic conductivity. However, it is possible to remove either θ or h by adopting the concept of differential water capacity, defined as the derivative of the soil water retention curve [19]:

$$C(h) = \frac{d\theta}{dh}. \tag{6}$$

The h-based formulation of Richards' equation is thus obtained by replacing Eq. (6) in Eq. (5) as:

$$C(h).\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) + \frac{\partial K}{\partial z}. \tag{7}$$

This is a essential equation in geotechnical and geo-environmental engineering and is used for modeling flow of water through unsaturated soils. For example, the two dimensional form of the equation can be used to model seepage in the unsaturated zone above water table in an earth dam.
Introducing a new term D, pore water diffusivity, defined as the ratio of the hydraulic conductivity to the differential water capacity, the θ-based form of Richards' equation may be obtained. D can therefore be written as:

$$D = \frac{K}{C} = \frac{K}{\frac{d\theta}{dh}} = K\frac{dh}{d\theta}. \tag{8}$$

Since D and K are highly dependent on water content. Combining (8) with Eq. (5) gives Richards' equation as:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}D\left(\frac{\partial \theta}{\partial z}\right) + \frac{\partial K}{\partial z}. \tag{9}$$

In order to solve Eq. (9), one must first accurately address the task of estimate D and K, both of which are dependent on water content. Various models have been recommended for determining these parameters. The Van Genuchten model [20] and Brook's and Corey's model [21] are the more frequently used models. The Van Genuchten model uses mathematical relations to relate soil water pressure head with water content and unsaturated hydraulic conductivity, through a concept called "relative saturation rate". This model matches experimental data but its functional form is rather complicated and it is therefore difficult to execute it in most analytical solution schemes. Brooks and Corey's model on the other hand has a more clear-cut definition and is therefore adopted in the present investigation. By this model the following hydraulic conductivity and water diffusitivity equation are obtained after some considerations [22, 23, 24]:

$$\begin{aligned} D(\theta) &= D_0\,(n+1)\,\theta^n, n \geq 0, \\ K(\theta) &= K_0\theta^k, k \geq 1, \end{aligned} \tag{10}$$

where K_0, D_0 and k are constants representing soil properties such as pore-size distribution, particle size, etc. In this demonstration of D and K, θ is scaled between 0 and 1 and diffusivity is normalized so that for all values of n, $\int_0^1 D(\theta)d\theta = 1$ [19]. Eq. (10) suggests that conductivity may have linear, parabolic, cubic, etc. variation with water content, related with k values of $1, 2, 3$, etc., respectively. Many analytical and numerical solutions to Richards' equation exist based on Brook's and Corey's representation of D and K. Replacing $n = 0$ and $k = 2$ in $D(\theta)$ and $K(\theta)$ represented in Eq. (10) yields the classic Burgers' equation broadly studied by many researchers [19, 25]. The generalized Burgers' equation is also obtained for general values of k and n [25].

As seen before, the two independent variables in Eq. (9) are time and depth. By applying the traveling wave technique [26], instead of time and depth, a new variable which is a linear combination of them is found. Tangent-hyperbolic function is usually applied to solve these transform equations [27]. Therefore the θ- based Richards' equation in order of $(n, 1)$ is obtained as [26]:

$$\frac{\partial\theta}{\partial t} + \alpha\theta^n \frac{\partial\theta}{\partial z} - \frac{\partial^2\theta}{\partial z^2} = 0, \tag{11}$$

and its exact solution is given by [28]:

$$\theta(z, t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(A_1\left[z - A_2 t\right]\right)^{\frac{1}{n}}\right), \tag{12}$$

where

$$A_1 = -\frac{\alpha n + n|\alpha|}{4(1+n)}(n \neq 0), A_2 = \frac{\gamma\alpha}{(1+n)},$$

where γ is an arbitrary coefficient which is selected as 1 in the given study [29]. By assuming $t = 0$ in Eq. (12), the initial condition for (11) can be found. For different values of n, the θ-based Richards' equation, which has been focus of this research, is solved here by the mixture of new integral transform and homotopy perturbation method. The basic idea of this method will be explained in the next section.

3. THE BASIC IDEA OF NITHPM

To show the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), u(x,0) = h(x), u_t(x,0) = f(x), \tag{13}$$

where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents

the general nonlinear differential operator and $g(x,t)$ is the source term.
Taking the new integral transform on both sides of Eq. (13) and using differential property of it with above initial conditions, we get

$$K[u(x,t)] = v^4 K[g(x,t)] + vh(x) + v^3 f(x) - v^4 K\left[Ru(x,t) + Nu(x,t)\right]. \tag{14}$$

Now, applying inverse new integral transform on both sides of Eq. (14), we get

$$u(x,t) = G(x,t) - K^{-1}\left\{v^4 K\left[Ru(x,t) + Nu(x,t)\right]\right\}, \tag{15}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{16}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{17}$$

for some He's polynomials $H_n(u)$ (see [30, 31]) that are given by

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0, 1, 2, 3, \cdots. \tag{18}$$

Substituting Eq. (16) and Eq. (17) in Eq. (15), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left\{K^{-1}\left[v^4 K\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right\}, \tag{19}$$

which is the combination of the new integral transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$\begin{aligned}
&p^0 : u_0(x,t) = G(x,t),\\
&p^1 : u_1(x,t) = -K^{-1}\left\{v^4 K\left[Ru_0(x,t) + H_0(u)\right]\right\},\\
&p^2 : u_2(x,t) = -K^{-1}\left\{v^4 K\left[Ru_1(x,t) + H_1(u)\right]\right\},\\
&\cdots.
\end{aligned}$$

Thus the series solution of Eq. (13) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots. \tag{20}$$

4. APPLICATION

In this section, while dealing with the solution of the Richards' equation based on the Brooks and Corey model, the mixture of new integral transform and homotopy perturbation method is discussed. Although this method has the capability to solve Eq. (9) for any given values of n. In the present analysis, only two different cases of n are considered for the sake of simplicity and conciseness.

Case 1. *The equation derived below is obtained by assuming* $n = 1$ *in Eq.* (11)

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \theta\frac{\partial\theta}{\partial z}, \tag{21}$$

with the initial condition

$$\theta(z,0) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{4}\right)\right). \tag{22}$$

As discussed in previous section, the NITHPM yields

$$\theta(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{4}\right)\right) + K^{-1}\left\{v^2 K\left[\frac{\partial^2\theta}{\partial z^2} - \theta\frac{\partial\theta}{\partial z}\right]\right\}. \tag{23}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n\theta_n(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{4}\right)\right) + pK^{-1}\left\{v^2 K\left[\sum_{n=0}^{\infty} p^n\theta_{n zz}(z,t) - \sum_{n=0}^{\infty} p^n H_n(\theta)\right]\right\}. \tag{24}$$

where $H_n(\theta)$ are He's polynomials that represent the nonlinear terms. The first few components of He's polynomials are given by:

$$H_0(\theta) = \theta_0\frac{\partial\theta_0}{\partial z},$$
$$H_1(\theta) = \theta_0\frac{\partial\theta_1}{\partial z} + \theta_1\frac{\partial\theta_0}{\partial z},$$
$$\cdots.$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 : \theta_0(z,t) &= \frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{4}\right), \\ p^1 : \theta_1(z,t) &= K^{-1}\left\{v^2 K\left[\theta_{0zz} - H_0(\theta)\right]\right\}, \\ &= \frac{1}{16}\operatorname{sech}^2\left(\frac{z}{4}\right)t, \\ p^2 : \theta_2(z,t) &= K^{-1}\left\{v^2 K\left[\theta_{1zz} - H_1(\theta)\right]\right\}, \\ &= \frac{1}{128}\operatorname{sech}^2\left(\frac{z}{4}\right)\tanh\left(\frac{z}{4}\right)t^2, \\ &\cdots . \end{aligned}$$

Thus the approximate solution of (11) for the case $n = 1$ is given by:

$$\theta(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(-\frac{z}{4}\right)\right) + \frac{1}{16}\operatorname{sech}^2\left(\frac{z}{4}\right)t + \frac{1}{128}\operatorname{sech}^2\left(\frac{z}{4}\right)\tanh\left(\frac{z}{4}\right)t^2 + \cdots . \quad (25)$$

From the Figure 1, we can conclude that moisture content $\theta(z,t)$

TABLE 1. Values of Moisture content $\theta(z,t)$ at different depth (z) and time (t) for $n = 1$

	Moisture Content $\theta(z,t)$						
z	$t = 0$	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$	$t = 2.5$	$t = 3$
0	0.5	0.53125	0.5625	0.59375	0.625	0.65625	0.6875
1	0.377451	0.407366	0.43809	0.469714	0.502237	0.53566	0.569981
2	0.268941	0.294228	0.320934	0.349059	0.378605	0.40957	0.441954
3	0.182426	0.201809	0.222672	0.245016	0.26884	0.294144	0.320928
4	0.119203	0.132952	0.14795	0.164198	0.181695	0.200442	0.220438
5	0.0758582	0.0850857	0.095242	0.106328	0.118344	0.131288	0.145161

decreases when depth z increases for fix time t and it increases when time t increases for fix depth z, which we can see from Figure 2. Its numerical representation shows in Table 1. Figure 3 demonstrate the approximate solution of (11) for $n = 1$. The comparison of numerical results obtained with $z = 0, 1, 2, 3, 4, 5$ for $t = 1, 3$ by homotopy analysis method (HAM), homotopy perturbation method (HPM), differential transform method (DTM) and Elzaki transform homotopy perturbation method (ETHPM) are shown in Table 2 & 3 and its graphical representation is given in Figures 4 & 5.

Case 2. *The equation derived below is obtained by assuming* $n = 2$ *in* (11) *as*

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \theta^2\frac{\partial\theta}{\partial z}, \quad (26)$$

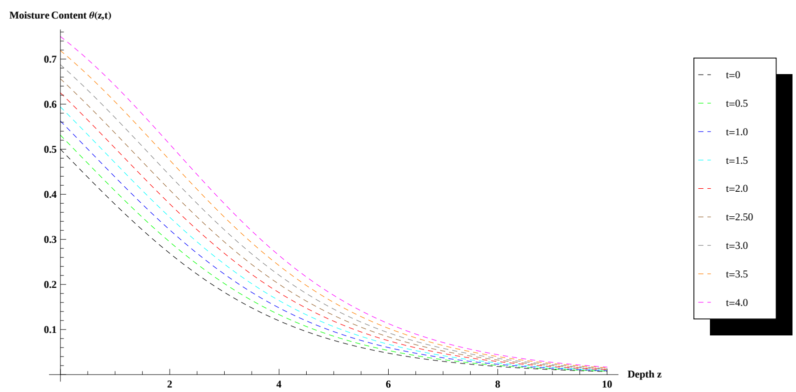

FIGURE 1. Moisture Content $\theta(z,t)$ versus Depth z for $n = 1$

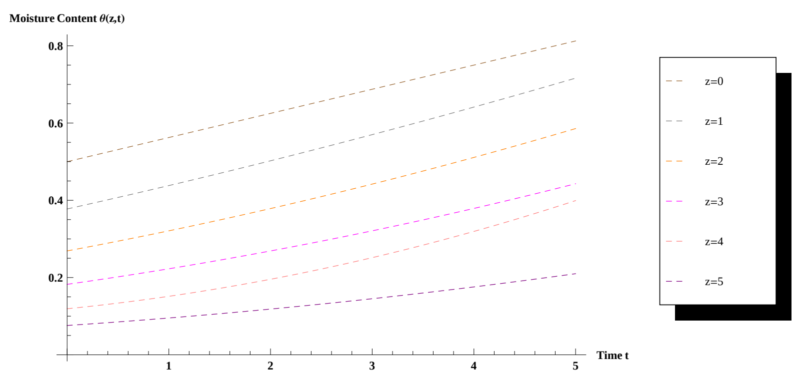

FIGURE 2. Moisture Content $\theta(z,t)$ versus Time t for $n = 1$

with the initial condition

$$\theta(z,0) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{3}\right)\right)^{\frac{1}{2}}.$$

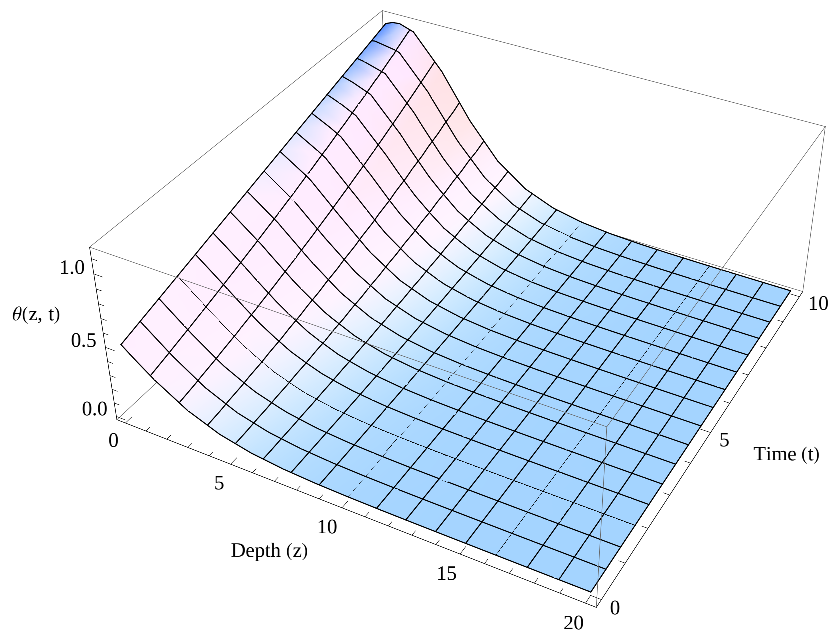

FIGURE 3. Approximate Solution of (11) by NITHPM when n=1

TABLE 2. Comparison between the results obtained by different solutions for $n = 1$ and $t = 1$

	Moisture Content $\theta(z,t)$					
z	$t = 1$					
	HAM	HPM	DTM	ETHPM	NITHPM	Exact
0	0.56217	0.5625	0.5625	0.5621	0.5625	0.5622
1	0.4378	0.4380	0.4382	0.4379	0.43809	0.4378
2	0.3208	0.3209	0.3211	0.3209	0.320934	0.3208
3	0.2227	0.2226	0.2228	0.2227	0.222672	0.2227
4	0.1480	0.1479	0.1480	0.1480	0.14795	0.1480
5	0.0953	0.0952	0.0953	0.0953	0.0953	0.0953

Using the same concept applying for Case 1, we get,

$$p^0 : \theta_0(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{3}\right)\right)^{\frac{1}{2}},$$

$$p^1 : \theta_1(z,t) = K^{-1}\left\{v^2 K\left[\theta_{0zz} - H_0(\theta)\right]\right\}$$

$$= \frac{1}{18}\text{Sech}\left(\frac{z}{3}\right)^2\left(2 - 2\tanh\left(\frac{z}{3}\right)\right)^{-\frac{1}{2}} t,$$

$$\cdots$$

TABLE 3. Comparison between the results obtained by different solutions for $n = 1$ and $t = 3$

	Moisture Content $\theta(z,t)$					
	t=3					
z	HAM	HPM	DTM	ETHPM	NITHPM	Exact
0	0.6791	0.6875	0.6875	0.6787	0.6875	0.6791
1	0.5621	0.5699	0.5737	0.5665	0.569981	0.5621
2	0.4378	0.4419	0.4469	0.4430	0.441954	0.4378
3	0.3208	0.3209	0.3249	0.3238	0.320928	0.3208
4	0.2227	0.2204	0.2227	0.2233	0.220438	0.2227
5	0.1480	0.1451	0.1463	0.1474	0.145161	0.1480

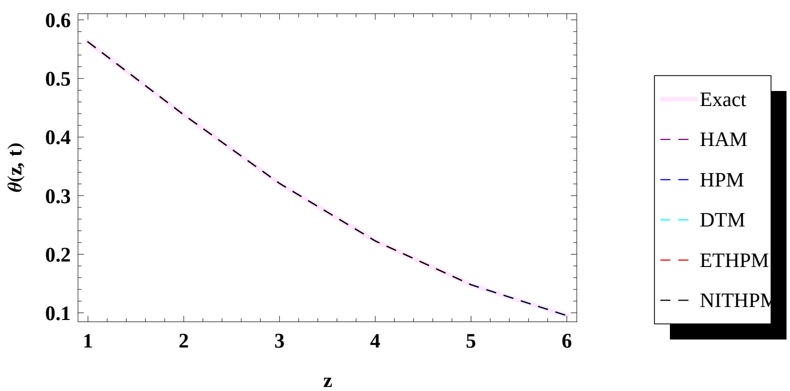

FIGURE 4. Comparison between the solution obtained by different methods and exact solution ($n = 1, t = 1$)

Then the approximate solution can be written as:

$$\theta(z,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-z}{3}\right)\right)^{\frac{1}{2}} + \frac{1}{18}\text{Sech}\left(\frac{z}{3}\right)^2\left(2 - 2\tanh\left(\frac{z}{3}\right)\right)^{-\frac{1}{2}} t + \cdots .$$

From the Figure 6, we can conclude that moisture content $\theta(z,t)$ decreases when depth z increases for fix time t and it increases when time t increases for fix depth z, which we can see from Figure 7. Its numerical representation shows in Table 4. Figure 8 demonstrate the approximate solution of (11) for $n = 2$. The comparison of numerical results obtained with $z = 0, 1, 2, 3, 4, 5$ for $t = 1$ by homotopy analysis method (HAM), homotopy perturbation method (HPM), differential transform method (DTM) and Elzaki transform

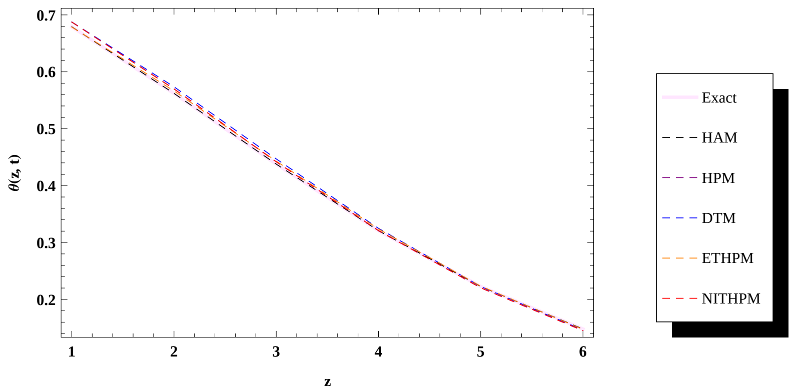

FIGURE 5. Comparison between the solution obtained by different methods and exact solution ($n = 1, t = 3$)

TABLE 4. Values of Moisture content $\theta(z,t)$ at different depth (z) and time (t) for $n = 2$

	Moisture Content $\theta(z,t)$					
z	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
0	0.707107	0.74639	0.785674	0.824958	0.864242	0.903525
1	0.582446	0.625208	0.66797	0.710731	0.753493	0.796265
2	0.456737	0.496899	0.537061	0.577223	0.617385	0.657547
3	0.345258	0.379047	0.412836	0.446625	0.480414	0.514203
4	0.254891	0.281372	0.307853	0.334334	0.360815	0.387296
5	0.185594	0.205505	0.225417	0.245328	0.265239	0.28515

homotopy perturbation method (ETHPM) are shown in Table 5 and its graphical representation is given in Figure 9.

TABLE 5. Comparison between the results obtained by different solutions for $n = 2$ and $t = 1$

	Moisture Content $\theta(z,t)$				
	$t = 1$				
z	HAM	HPM	DTM	ETHPM	NITHPM
0	0.7452	0.7453	0.7452	0.7464	0.74639
1	0.625	0.6252	0.6252	0.6252	0.625208
2	0.4977	0.4977	0.4979	0.4969	0.496899
3	0.3802	0.3803	0.3803	0.379	0.379047
4	0.2826	0.2826	0.2826	0.2814	0.281372
5	0.2065	0.2065	0.2065	0.2055	0.205505

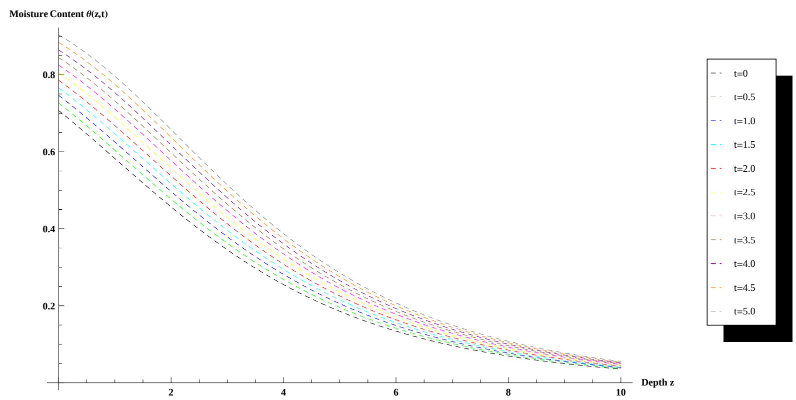

FIGURE 6. Moisture Content $\theta(z,t)$ versus Depth z for $n = 2$

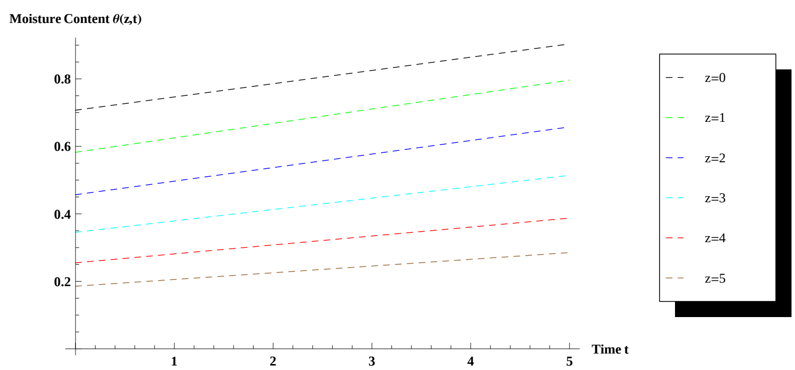

FIGURE 7. Moisture Content $\theta(z,t)$ versus Time t for $n = 2$

4. CONCLUDING REMARKS

The conclusion that can be draw from our results is that the combination of the new integral transform and homotopy perturbation method is an effective tool to deal with nonlinear Richards' equation and provides an accurate approximation. Illustrative examples proved the high accuracy of the results obtained using mixture of new integral transform and homotopy perturbation method. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result.

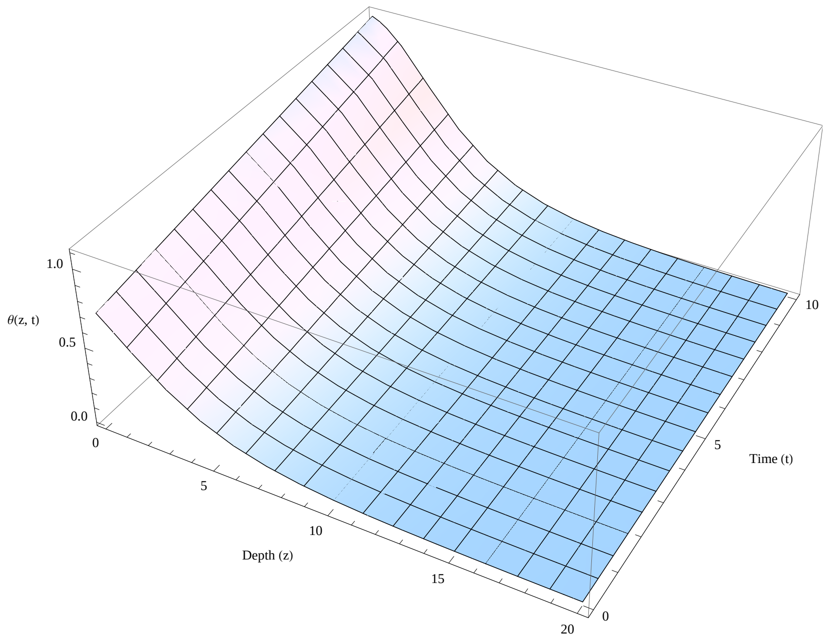

FIGURE 8. Approximate Solution of (11) by NITHPM when $n = 2$

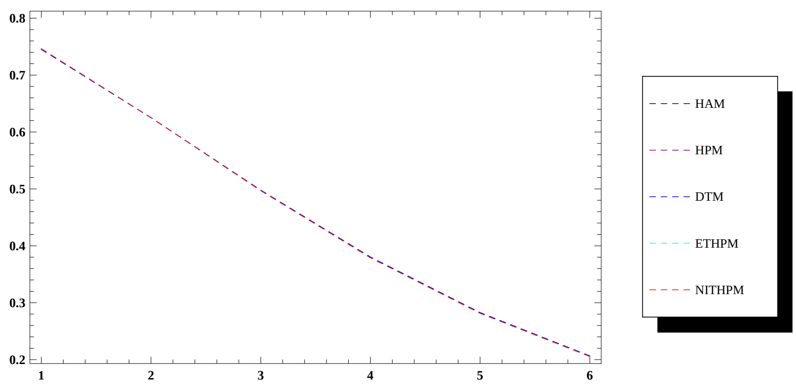

FIGURE 9. Comparison between the solution obtained by different methods for $(n = 2, t = 1)$

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