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GENERALIZED ERROR ESTIMATION OF THE TAU METHOD IN ORDINARY DIFFERENTIAL EQUATIONS

K. ISSA[1], R. B. ADENIYI AND B. M. YISA

ABSTRACT. The numerical method for solving Non-linear ODEs is reported here based on the generalized Tau approximant earlier derived. The error estimation of the Tau method for both linear and non-linear ODEs, overdetermined and non-overdetermined type were also examined and generalized with the use of chebyshev polynomials as perturbation terms. Some numerical problems were solved to illustrate the effectiveness and simplicity of the generalized scheme.

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1. INTRODUCTION

The Tau method [3, 4, 7, 14, 16, 17, 18, 21, 26, 28] for the case of numerical solution of ODEs and in [5, 8, 22] for the case of numerical solution of PDEs. Application of orthogonal polynomials especially chebyshev and legendre polynomials and their numerical merits in solving ODEs and PDEs numerically have been discussed in series of papers [3, 6, 17, 18, 29]. The idea of these orthogonal polynomials as perturbation terms is the central idea and philosophical feature of the Tau method. Ortiz and Pham Ngoc [23] worked on the structural relations between the Tau method and other numerical methods [13, 23].
The error estimation for ODEs has been reported by many researchers, including [1, 2, 3, 4, 7, 9, 10, 15, 19, 26, 27, 28, 30] to mention view. The dependence of error on the degree and on the length of the interval of approximation was discussed by El-Daou and Ortiz [12].

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www.nigerianmathematicalsociety.org
[1]Corresponding author

2. REVIEW OF GENERALIZED TAU APPROXIMANT

In this section, we consider Tau approximant for the recursive form of the differential equation:

$$Ly(x) \equiv \sum_{r=0}^{m} \left(\sum_{k=0}^{N_r} P_{r,k} x^k \right) y^{(r)}(x) = \sum_{r=0}^{\sigma} f_r x^r, \; a \leq x \leq b \tag{1}$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \; k = 1(1)m \tag{2}$$

by seeking an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \; r < +\infty \tag{3}$$

of $y(x)$ which is the exact solution of the corresponding perturbed system

$$Ly_n(x) = \sum_{r=0}^{\sigma} f_r x^r + H_n(x) \tag{4}$$

$$L^* y_n(x_{rk}) = \alpha_k, \; k = 1(1)m \tag{5}$$

where L is the linear differential operator, $y^{(r)}(x)$ is the derivative of order r of $y(x)$,$P_{r,k}(x)$, α_k, x_{rk}, a and b are real constants, σ is an integer,

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} x^r \tag{6}$$

is the perturbation term, the parameters $\tau_i, \; i = 1(1)(m+s)$ in (6) are to be determined and

$$T_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \equiv Cos \left[nCos^{-1} \left(\frac{2x-a-b}{b-a} \right) \right] \tag{7}$$

is the chebyshev polynomial valid in the interval $[a, b]$. The integer s is the number of overdetermination of (1), and is given by:

$$s = max \{ N_r - r > 0 \,|\, 0 \leq r \leq m \} \tag{8}$$

In 1956, Lanczos [20] introduced the use of canonical polynomial $\{Q_r(x)\}$, $r \geq 0$ into the Tau method and the difficulty involved in their construction was removed by Ortiz [19] and generalized by Yisa and Adeniyi [29] as

$$Q_n(x) = \frac{1}{\sum_{k=0}^{m} k!\binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\} \quad (9)$$

So, by varying m and s in equation (1), we have:

For Case $m = 1$ and $s = 1$

We have

$$\begin{aligned} Ly_n(x) &= \sum_{r=0}^{\sigma} f_r x^r + \sum_{i=0}^{1} \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} x^r \\ &= \sum_{r=0}^{\sigma} f_r LQ_r(x) + \sum_{i=0}^{1} \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} LQ_r(x) \\ &= L\left\{ \sum_{r=0}^{\sigma} f_r Q_r(x) + \sum_{i=0}^{1} \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} Q_r(x) \right\} \end{aligned}$$

$$y_n(x) = \sum_{r=0}^{\sigma} f_r Q_r(x) + \tau_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(x) + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} Q_r(x) \quad (10)$$

Now, substituting (9) in (10); expanding the resulting equation and equate the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) to zero, we obtain

$$\tau_1 \sum_{r=0}^{n} C_r^{(n)} P_r + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \quad (11)$$

Take $P_0 = 1$. Equation (11) is the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) and for ease generalization we assume $Q_0(x) = 1$, we obtain

$$y_n(x) = \sum_{r=1}^{\sigma} f_r q_r(x) + \sum_{i=0}^{1} \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n+i)} q_r(x) \quad (12)$$

where $q_n = Q_n(x) - P_n$ and

$$P_n = \frac{-P_{0,0}P_{n-1} - (n-1)P_{1,0}P_{n-2} - (n-1)P_{1,1}P_{n-1}}{P_{0,1} + (n-1)P_{1,2}} \quad (13)$$

Case $m = 1, \; s = 2$
From (4), we have

$$y_n(x) = \sum_{r=0}^{\sigma} f_r Q_r(x) + \tau_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(x) + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} Q_r(x) + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} Q_r(x) \tag{14}$$

Again, putting (9) in (14); expanding the resulting equation and equate the coefficient of undetermined canonical polynomials (i.e $Q_0(x)$, $Q_1(x)$) to zero, we obtain

$$\tau_1 \sum_{r=0}^{n} C_r^{(n)} P_r + \tau_2 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \tau_3 \sum_{r=0}^{n+2} C_r^{(n+2)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \tag{15}$$

Take $P_0 = 1$, $P_1 = 0$ for coefficient of $Q_0(x)$, for coefficient of $Q_1(x)$ take $P_0 = 0$, $P_1 = 1$ and for $y_n(x)$ we assume $Q_0(x) = Q_1(x) = 1$,and take $P_0 = P_1 = 1$, we obtain

$$y_n(x) = \sum_{r=2}^{\sigma} f_r q_r(x) + \sum_{i=0}^{2} \tau_{i+1} \sum_{r=2}^{n+i} C_r^{(n+i)} q_r(x) \tag{16}$$

where

$$P_n = \frac{-(n-2)P_{1,0}P_{n-3} - (P_{0,0} + (n-2)P_{1,1})\, P_{n-2} - (P_{0,1} + (n-2)P_{1,2})\, P_{n-1}}{P_{0,2} + (n-2)P_{1,3}} \tag{17}$$

and $q_n = Q_n(x) - P_n$
Case $m = 2, \; s = 1$:
substituting (9) in (4); expanding the resulting equation and equate the coefficient of undetermined canonical polynomial (i.e $Q_0(x)$) to zero, we obtain

$$\tau_1 \sum_{r=0}^{n-1} C_r^{(n-1)} P_r + \tau_2 \sum_{r=0}^{n} C_r^{(n)} P_r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \tag{18}$$

Take $P_0 = 1$ for the coefficient of $Q_0(x)$, and for $y_n(x)$ we assume $Q_0(x) = P_0 = 1$, then we obtain the solution:

$$y_n(x) = \sum_{r=1}^{\sigma} f_r q_r(x) + \sum_{i=0}^{2} \tau_{i+1} \sum_{r=1}^{n-1+i} C_r^{(n-1+i)} q_r(x) \tag{19}$$

and

$$y_n'(x) = \sum_{r=1}^{\sigma} f_r Q_r'(x) + \sum_{i=0}^{2} \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n-1+i)} Q_r'(x)$$

Case $m = 2,\ s = 2$**:**
substituting (9) in (4); expanding the resulting equation and equate the coefficient of undetermined canonical polynomials (i.e $Q_0(x)$ and $Q_1(x)$) to zero, we obtain

$$\tau_1 \sum_{r=0}^{n-1} C_r^{(n-1)} P_r + \tau_2 \sum_{r=0}^{n} C_r^{(n)} P_r + \tau_3 \sum_{r=0}^{n+1} C_r^{(n+1)} P_r + \tau_4 \sum_{r=0}^{n} C_r^{(n+2)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \quad (20)$$

for the coefficient of $Q_0(x)$, take $P_0 = 1$ and $P_1 = 0$, and for the coefficient of $Q_1(x)$, take $P_1 = 1$, $P_0 = 0$ and for $y_n(x)$ we assume $Q_0(x) = Q_1(x) = P_0 = P_1 = 1$, then we obtain the solution

$$y_n(x) = \sum_{r=2}^{\sigma} f_r q_r(x) + \sum_{i=0}^{3} \tau_{i+1} \sum_{r=1}^{n-1+i} C_r^{(n-1+i)} q_r(x) \quad (21)$$

and

$$y_n'(x) = \sum_{r=2}^{\sigma} f_r Q_r'(x) + \sum_{i=0}^{3} \tau_{i+1} \sum_{r=1}^{n+i} C_r^{(n-1+i)} Q_r'(x)$$

Observing the trend in the Tau approximant of the first and second order ordinary differential equations presented above, we derived the general formula

$$y_n(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x) \quad (22)$$

Assume $Q_r(x) = P_r = 1, r = 0(1)(s-1)$

$$y_n^{\lambda}(x) = \sum_{r=s}^{\sigma} f_r Q_r^{\lambda}(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} Q_r^{\lambda}(x) = \alpha_\lambda, \lambda = 0(1)(m-1) \quad (23)$$

$$\sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \quad (24)$$

where $q_r(x) = Q_r(x) - P_r$ and equation (24) is the coefficient of undetermined canonical polynomials and

$$P_r = \begin{cases} \frac{-1}{\sum_{k=0}^{m} k!\binom{r-s}{k} P_{k,k+s}} \left\{ \sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\binom{r-s}{j} P_{j,j-k} \right) P_{r-s-k} \right. \\ \left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\binom{r-s}{j} P_{j,j+k} \right) P_{r-s+k} \right\}, r \geq s \\ 1, \text{when equating the coefficient of } Q_r(x) \text{ to zero, } \forall r = 0, \ldots, (s-1) \\ 0, \text{otherwise}, \forall r = 0, \ldots, (s-1) \end{cases} \quad (25)$$

Mathematica code for P_r:

$$P_r = \frac{1}{\sum_{k=0}^{m} k!\text{Binomial}[r-s,k]P_{k,k+s}} \left(-\left\{ \sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\text{Binomial}[r-s,j]P_{j,j-k} \right) P_{r-s-k} + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\text{Binomial}[r-s,j]P_{j,j+k} \right) P_{r-s+k} \right\} \right)$$

MATLAB program for P_r

```
P(1)=1;%this is P(0)
P(2)=0;
pp=input('enter p as a matrix >>');
n=input('enter n>>');
m=input('enter m>>');
s=input('enter s>>');


A=0;
for k=0:m;
 A=A+factorial(k)*nchoosek(n-s,k)*pp(k+1,k+s+1);
end
B=0;
for k=1:m;
 for j=k:m
  B=B+factorial(j)*nchoosek(n-s,j)*pp(j+1,j-k+1)*...
  P(n-s-k+1);
 end
end
C=0;
for k=0:s-1;
 for j=0:m;
  C=C+vpa(factorial(j)*nchoosek(n-s,j)*pp(j+1,j+k+1)*...
  P(n-s+k+1));

 end
end
 P(n+1)=-1/A*(B+C);
```

The validity of P_r and $Q_r(x)$ have been established by [17] and [29] respectively.

3. EXTENSION TO NON-LINEAR PROBLEMS

In this section, we shall consider an extension of the generalized recursive formulation of the Tau method to non-linear problems. For this purpose, we shall employ Newton-Kantorovich linearization process [11, 18, 24, 25] to non-linear differential equation of the form:

$$G(x, y(x), y'(x), \dots, y^{(m)}(x)) = \sum_{r=0}^{\sigma} f_r x^r \tag{26}$$

and the process of Newton-Kantorovich linearization, derived from the Taylor series expansion in several variables of G, is given by:

$$G + \triangle y \frac{\partial G}{\partial y} + \triangle y' \frac{\partial G}{\partial y'} + \triangle y'' \frac{\partial G}{\partial y''} + \cdots + \triangle y^{(m)} \frac{\partial G}{\partial y^{(m)}} = \sum_{r=0}^{\sigma} f_r x^r \tag{27}$$

where $\triangle y_k^i = y_{k+1}^i - y_k^i$, $i = 0, 1 \dots, m$
We seek $k - th$ iterative approximant of the form:

$$y_{n,k}(x) = \sum_{r=0}^{n} a_{r,k} x^r \tag{28}$$

Adding the perturbation terms to equation (27) we have:

$$\sum_{j=0}^{m} (y_{n,k+1}^j(x) - y_{n,k}^j(x)) \frac{\partial G_{n,k}}{\partial y_{n,k}^j} = \sum_{r=0}^{\sigma} f_r x^r - G_{n,k} + H_{n,k}(x) \tag{29}$$

where

$$G_{n,k} \equiv G(x, y_{n,k}(x), y'_{n,k}(x), \dots, y_{n,k}^{(m)}(x)),\ k = 0, 1, \dots \tag{30}$$

The number of overdetermination s, for the non-linear problems, unlike in the case of linear problems, depends on n and can be very large depending on the degree of non-linearity of the problem under consideration. The iterative process is repeated until

$$| \xi_{n,k} - \xi_{n,k+1} | \leq \text{Tolerance Value}$$

where

$$\xi_{n,k} = \max\{| y_k(x) - y_{n,k}(x) | : a \leq x \leq b\} \tag{31}$$

The iterative scheme needs a suitable choice of an initial approximation $y_{n,0}(x)$ from linearization process, for a rapid convergence.

In most problems, the initial approximation $y_{n,0}(x)$ is taken to be the simplest polynomial satisfying the associated condition in (2). In some cases, good choice of initial approximation is obtained from the given differential equation itself (see Example 3.1 below).

3.1. NUMERICAL PROBLEMS

We shall consider here three problems of interest for the illustration of the method of the preceding section. It is to be noted that the results presented in the tables below were obtained using mathematica 7.0 package.
We illustrate the accuracy of the method presented in this work by computing $\xi_{n,k}$ (maximum error) for some numerical Problems.

Problem 3.1[18]: Consider the first order non-linear problem

$$y'(x) + (2x-1)y^2(x) = 0 \tag{32}$$

subject to initial condition

$$y(0) = 1,\ 0 \le x \le 1 \tag{33}$$

with exact solution $y(x) = (x^2 - x + 1)^{-1}$
For this problem, we have

$$G_k \equiv G(x, y_k(x), y'_k(x)) = y'_k(x) + (2x-1)y_k^2(x)$$

so that

$$G_k + \triangle y_k \frac{\partial G_k}{\partial y_k} + \triangle y'_k \frac{\partial G_k}{\partial y'_k} = 0 \tag{34}$$

where

$$\triangle y_k = y_{k+1} - y_k,\ \triangle y'_k = y'_{k+1} - y'_k,\ \frac{\partial G_k}{\partial y_k} = 2y_k(2x-1),\ \frac{\partial G_k}{\partial y'_k} = 1 \tag{35}$$

substituting (35) in (34), we have:

$$y'_{k+1} + (4x-2)y_k y_{k+1} = (2x-1)y_k^2 \tag{36}$$

From (29), we have

$$y'_{n,k+1} + (4x-2)y_{n,k}y_{n,k+1} = (2x-1)y_{n,k}^2 + H_{n,k}(x)$$

where

$$H_{n,k}(x) = \sum_{i=0}^{s} \tau_{i+1,n} T_{n+i,k}(x)$$

Thus, the sequence of linearized Tau problem to be solved is:

$$y'_{n,k+1} + (4x-2)y_{n,k}y_{n,k+1} = (2x-1)y^2_{n,k} + \sum_{i=0}^{s} \tau_{i+1,n}T_{n+i,k}(x) \quad (37)$$

$$y_{n,k+1}(0) = 1, k = 0, 1, 2, \ldots \quad (38)$$

For the choice of initial approximation, we have from (33), $y'(0) = 1$ (since $y(0) = 1$). Hence , if we assume an initial approximation of the form $y = a + bx$, then we get $y(x) = x + 1$ by using $y(0) = 1$ and $y'(0) = 1$ to determine a and b. So, we choose $y_{n,0}(x) = x + 1$ and then compute the approximant solution.

First Iteration (k=0): we have from (36)

$$y'_1(x) + (4x^2 + 2x - 2)y_1(x) = 2x^3 + 3x^2 - 1 \quad (39)$$

$$y'_1(0) = 1$$

Comparing (39) with (1), we have $m = 1$, $s = 2$, and the perturbed problem (37) becomes

$$y'_{n,1}(x) + (4x^2 + 2x - 2)y_{n,1}(x) = 2x^3 + 3x^2 - 1 + \tau_1 T_n(x) + \tau_2 T_{n+1}(x) + \tau_3 T_{n+2}(x)$$

Applying (22 and 24), for the solution of degree 5, 6 and 7, we have

$$y_{5,1}(x) = 1 + \frac{13734241x}{14036168} + \frac{2477864x^2}{5263563} - \frac{15478136x^3}{5263563} + \frac{2400320x^4}{1754521} + \frac{1142176x^5}{5263563}$$

$$\begin{aligned} y_{6,1}(x) =& 1 + \frac{758745347x}{754661830} - \frac{1599024368x^2}{7923949215} + \frac{923988736x^3}{1584789843} - \frac{15803064192x^4}{2641316405} \\ &+ \frac{54946095104x^5}{7923949215} - \frac{5926750208x^6}{2641316405} \end{aligned}$$

$$\begin{aligned} y_{7,1}(x) =& 1 + \frac{109326648889x}{109205132560} - \frac{643141379x^2}{13650641570} - \frac{1645570879x^3}{2730128314} - \frac{15100542224x^4}{6825320785} \\ &+ \frac{7700535256x^5}{6825320785} + \frac{14043641024x^6}{6825320785} - \frac{1679799488x^7}{1365064157} \end{aligned}$$

Second Iteration (k=1): We similarly obtain

$$y_{5,2}(x) = 1+0.982633448x+0.398369665x^2-2.746545092x^3+1.345127453x^4+0.020877660x^5$$

$$\begin{aligned} y_{6,2}(x) =& 1 + 1.001540930x - 0.073718784x^2 - 0.27552491x^3 - 3.801892850x^4 \\ &+ 4.714527741x^5 - 1.564467919x^6 \end{aligned}$$

$$\begin{aligned} y_{7,2}(x) =& 1 + 1.001763174x - 0.081717596x^2 - 0.214195149x^3 - 3.996928214x^4 \\ &+ 5.014854129x^5 - 1.786985423x^6 + 0.0636669636x^7 \end{aligned}$$

and for **Third Iteration (k=2)** we have

$$y_{5,3}(x) = 1 + 0.982318x + 0.405095x^2 - 2.77481x^3 + 1.3874x^4 + 5.28099 \times 10^{-6}x^5$$

$$y_{6,3}(x) = 1 + 1.001699849x - 0.078836193x^2 - 0.241731719x^3 - 3.889122757x^4 + 4.811985927x^5 - 1.603995100x^6$$

$$y_{7,3}(x) = 1 + 1.001699864x - 0.0788367194x^2 - 0.241727690x^3 - 3.889135527x^4 + 4.812005511x^5 + 4.111601230 \times 10^{-6}x^7$$

The Table below gives the maximum error for each iteration

Table 3.1:Maximum Error for Example 4.1

Iteration (k)	Degree 5 ($\xi_{5,k}$)	Degree 6 ($\xi_{6,k}$)	Degree 7 ($\xi_{7,k}$)
$k = 0$	9.37×10^{-2}	9.39×10^{-2}	9.41×10^{-2}
$k = 1$	2.23×10^{-3}	4.64×10^{-4}	4.44×10^{-4}
$k = 2$	2.27×10^{-3}	1.67×10^{-4}	1.70×10^{-4}

Problem 3.2(A constant coefficients second order Homogeneous Problem): $y''(x) - y(x)y'(x) = 0$
together with the given conditions

$$y(0) = 0,\ y(1) = \tanh(\frac{-1}{2})$$

with analytic solution $y(x) = \tanh(\frac{-x}{2})$, $0 \leq x \leq 1$
After linearized, we have

$$y''_{k+1} - y_k y'_{k+1} - y'_k y_{k+1} = -y_k y'_k$$

$$y_{k+1}(0) = 0,\ y_{k+1} = \tanh(\frac{-1}{2}), 0 \leq x \leq 1$$

see Table 3.2 for the maximum error

Problem 3.3 Consider non-homogeneous Problem:

$$y'(x) - (y(x))^2 = 1$$

$$y(0) = 1$$

with analytic solution $y(x) = \tan x$, $0 \leq x \leq \frac{\pi}{4}$
This leads to the linearized problem

$$y'_{k+1} - 2y_k y_{k+1} = 1 - y_k^2$$

$$y_{k+1}(0) = 0$$

see Table 3.3 for the maximum error

Table 3.2:Maximum Error for Example 3.2

Iteration (k)	$\xi_{5,k}$	$\xi_{6,k}$	$\xi_{7,k}$
$k=0$	1.14×10^{-5}	9.22×10^{-6}	9.42×10^{-6}
$k=1$	5.91×10^{-6}	2.64×10^{-7}	3.95×10^{-8}
$k=2$	5.91×10^{-6}	2.64×10^{-7}	3.99×10^{-8}

Table 3.3:Maximum Error for Example 3.3

Iteration (k)	$\xi_{5,k}$	$\xi_{6,k}$	$\xi_{7,k}$
$k=0$	5.18×10^{-3}	5.17×10^{-3}	5.16×10^{-3}
$k=1$	1.31×10^{-4}	2.86×10^{-5}	3.77×10^{-6}
$k=2$	1.31×10^{-4}	2.39×10^{-5}	3.78×10^{-6}

3.2. GENERALIZATION OF ERROR ESTIMATION FOR THE TAU METHOD

In this section, we shall derive the general error estimation for the recursive form of the Tau method using polynomial economization process. Adeniyi [2, 3] reported a polynomial error estimate $(e_n(x))_{n+1} \cong E_{n+1}(x)$ of degree $(n+1)$ as:

$$(e_n(x))_{n+1} \cong E_{n+1}(x) = \frac{\phi_n \nu_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{\phi_n \nu_m(x) \sum_{r=0}^{n-m+1} C_r^{(n-m+1)} x^r}{C_{n-m+1}^{(n-m+1)}} \tag{40}$$

where $\nu_m(x)$ is to ensure that $E_{n+1}(x)$ satisfies some or all the homogeneous conditions of $e_n(x)$.
He also reported that:

$$L\left(E_{n+1}(x)\right) = \hat{H}_{n+1}(x) - H_n(x) \tag{41}$$

where

$$\hat{H}_{n+1}(x) = \sum_{i=0}^{m+s} \tau_{i+1} \sum_{r=0}^{n-m+i+2} C_r^{(n-m+i+2)} x^r \tag{42}$$

After assuming that a transformation has been made such that $a=0$, $b=1$ (40) becomes:

$$E_{n+1}(x) = \frac{\phi_n(x) \sum_{r=0}^{n-m+1} C_r^{(n-m+1)} x^{r+m}}{C_{n-m+1}^{(n-m+1)}} \tag{43}$$

Considering the differential equation (1) in cases (that is varying m and s).

Case $m=1$, $s=1$

$$Ly(x) \equiv \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right) y'(x) + \left(P_{0,0} + P_{0,1}x\right) y(x) = \sum_{r=0}^{\sigma} f_r x^r \tag{44}$$

The linear differential operator L of equation (44) is:

$$L = \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right)\frac{d}{dx} + (P_{0,0} + P_{0,1}x) \tag{45}$$

Applying differential operator L on $E_{n+1}(x)$, we have:

$$L\left(E_{n+1}(x)\right) = \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2\right)\frac{d}{dx}\left(E_{n+1}(x)\right) + (P_{0,0} + P_{0,1}x)\left(E_{n+1}(x)\right) \tag{46}$$

where

$$E_{n+1}(x) = \frac{\phi_n \sum_{r=0}^{n} C_r^{(n)} x^{r+1}}{C_n^{(n)}} \tag{47}$$

Substituting equation (47) in equation (46), we obatin:

$$\begin{aligned} L\left(E_{n+1}(x)\right) =& \frac{\phi_n}{C_n^{(n)}}\left[P_{1,0}\sum_{r=0}^{n}(r+1)C_r^{(n)}x^r + \sum_{r=0}^{n}\left(P_{0,0} + (r+1)P_{1,1}\right)C_r^{(n)}x^{r+1}\right.\\ &\left.+\sum_{r=0}^{n}\left(P_{0,1} + (r+1)P_{1,2}\right)C_r^{(n)}x^{r+2}\right] \end{aligned} \tag{48}$$

from equation (41), we obtain

$$L\left(E_{n+1}(x)\right) = \hat{\tau}_1\sum_{r=0}^{n+1}C_r^{(n+1)}x^r + \hat{\tau}_2\sum_{r=0}^{n+2}C_r^{(n+2)}x^r - \tau_1\sum_{r=0}^{n}C_r^{(n)}x^r - \tau_2\sum_{r=0}^{n+1}C_r^{(n+1)}x^r \tag{49}$$

substituting equation (49) in (48) and equate the corresponding coefficient of x^{n+i} , $i = 0, 1, \ldots, m + s$, we obatin:

$$\hat{\tau}_2 C_{n+2}^{(n+2)} = \frac{\phi_n}{C_n^{(n)}}\left[P_{0,1} + (n+1)P_{1,2}\right]C_n^{(n)} \tag{50}$$

$$\hat{\tau}_1 C_{n+1}^{(n+1)} + \hat{\tau}_2 C_{n+1}^{(n+2)} - \tau_2 C_{n+1}^{(n+1} = \frac{\phi_n}{C_n^{(n)}}\left[\left(P_{0,0} + (n+1)P_{1,1}\right)C_n^{(n)} + \left(P_{0,1} + nP_{1,2}\right)C_{n-1}^{(n)}\right] \tag{51}$$

$$\begin{aligned} &\hat{\tau}_1 C_n^{(n+1)} + \hat{\tau}_2 C_n^{(n+2)} - \tau_1 C_n^{(n)} - \tau_2 C_n^{(n+1} = \frac{\phi_n}{C_n^{(n)}}\left[(n+1)P_{1,0}C_n^{(n)} + \right.\\ &\left.\left(P_{0,0} + nP_{1,1}\right)C_{n-1}^{(n)} + \left(P_{0,1} + (n-1)P_{1,2}\right)C_{n-2}^{(n)}\right] \end{aligned} \tag{52}$$

From equations (50), (51) and (52), we obtain:

$$\phi_n = \frac{\tau_1 C_n^{(n)}}{\gamma_{1,1}} \tag{53}$$

where

$$\begin{aligned}\gamma_{1,1} = &\left(\frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}}{C_n^{(n)}}\right) P_{0,0} + \left(\frac{C_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{C_n^{(n+2)}}{C_{n+2}^{(n+2)}}\right.\\ &\left. - \frac{C_{n-2}^{(n)}}{C_n^{(n)}}\right) P_{0,1} - (n+1)P_{1,0} + \left(\frac{(n+1)C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{nC_{n-1}^{(n)}}{C_n^{(n)}}\right) P_{1,1} +\\ &\left(\frac{nC_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{(n+1)C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{(n+1)C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \right.\\ &\left.\frac{(n-1)C_{n-2}^{(n)}}{C_n^{(n)}}\right) P_{1,2}\end{aligned} \tag{54}$$

Case $m = 1, \;\; s = 2$

$$\begin{aligned} Ly(x) &\equiv \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2 + P_{1,3}x^3\right) y'(x) + \left(P_{0,0} + P_{0,1}x + P_{0,2}x^2\right) y(x)\\ &= \sum_{r=0}^{\sigma} f_r x^r \end{aligned} \tag{55}$$

The linear differential operator L of equation (55) is:

$$L = \left(P_{1,0} + P_{1,1}x + P_{1,2}x^2 + P_{1,3}x^3\right) \frac{d}{dx} + \left(P_{0,0} + P_{0,1}x + P_{0,2}x^2\right) \tag{56}$$

Applying differential operator L on $E_{n+1}(x)$, we have:

$$\begin{aligned} L\left(E_{n+1}(x)\right) = &\left(P_{1,0} + P_{1,1}x + P_{1,2}x^2 + P_{1,3}x^3\right) \frac{d}{dx} \left(E_{n+1}(x)\right)\\ &+ \left(P_{0,0} + P_{0,1}x + P_{0,2}x^2\right) \left(E_{n+1}(x)\right) \end{aligned} \tag{57}$$

where

$$L\left(E_{n+1}(x)\right) = \sum_{i=0}^{2} \hat{\tau}_{i+1} \sum_{r=0}^{n+i+1} C_r^{(n+i+1)} x^r - \sum_{i=0}^{2} \tau_{i+1} \sum_{r=0}^{n+i} C_r^{(n+i)} x^r \tag{58}$$

Substituting equation (47), (58) in equation (57) and equate the corresponding coefficient of x^{n+i}, $i = 0(1)3$, we obtain:

$$\hat{\tau}_3 C_{n+3}^{(n+3)} = \phi_n \left[P_{0,2} + (n+1)P_{1,3}\right] \tag{59}$$

$$\begin{aligned}\hat{\tau}_2 C_{n+2}^{(n+2)} + \hat{\tau}_3 C_{n+2}^{(n+3)} - \tau_3 C_{n+2}^{(n+2} = &\frac{\phi_n}{C_n^{(n)}} \Big[\left(P_{0,1} + (n+1)P_{1,2}\right) C_n^{(n)}\\ &+ \left(P_{0,2} + nP_{1,3}\right) C_{n-1}^{(n)}\Big]\end{aligned} \tag{60}$$

$$\begin{aligned}&\hat{\tau}_1 C_{n+1}^{(n+1)} + \hat{\tau}_2 C_{n+1}^{(n+2)} + \hat{\tau}_3 C_{n+1}^{(n+3)} - \tau_2 C_{n+1}^{(n+1)} - \tau_3 C_{n+1}^{(n+2} = \frac{\phi_n}{C_n^{(n)}} \left[(P_{0,0}\right.\\ &\left. + (n+1)P_{1,1}\right) C_n^{(n)} + \left(P_{0,1} + nP_{1,2}\right) C_{n-1}^{(n)} + \left(P_{0,2} + (n-2)P_{1,3}\right) C_{n-3}^{(n)}\Big]\end{aligned} \tag{61}$$

$$\begin{aligned}&\hat{\tau}_1 C_n^{(n+1)} + \hat{\tau}_2 C_n^{(n+2)} + \hat{\tau}_3 C_n^{(n+3)} - \tau_1 C_n^{(n)} - \tau_2 C_n^{(n+1)} - \tau_3 C_n^{(n+2}\\ &= \frac{\phi_n}{C_n^{(n)}} \Big[(n+1)P_{1,0} C_n^{(n)} + \left(P_{0,0} + nP_{1,1}\right) C_{n-1}^{(n)} + \left(P_{0,1} + (n-1)P_{1,2}\right) C_{n-2}^{(n)}\\ &+ \left(P_{0,2} + (n-2)P_{1,3}\right) C_{n-3}^{(n)}\Big]\end{aligned} \tag{62}$$

From (59), (60), (61) and (62), we obtain:

$$\phi_n = \frac{\tau_1 C_n^{(n)}}{\gamma_{1,2}} \tag{63}$$

where

$$\begin{aligned}
\gamma_{1,2} = &\left(\frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}}{C_n^{(n)}}\right) P_{0,0} + \left(\frac{C_n^{(n+1)}C_{n-1}^{(n)}}{C_n^{(n)}C_{n+1}^{(n+1)}} - \frac{C_n^{(n+1)}C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}} + \frac{C_n^{(n+2)}}{C_{n+2}^{(n+2)}}\right.\\
&\left.-\frac{C_{n-2}^{(n)}}{C_n^{(n)}}\right) P_{0,1} + \left(\frac{C_n^{(n+1)}C_{n-2}^{(n)}}{C_n^{(n)}C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}C_n^{(n+1)}C_{n+1}^{(n+2)}}{C_n^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}} + \frac{C_n^{(n+1)}C_{n+1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}\right.\\
&\left.-\frac{C_n^{(n+1)}C_{n+1}^{(n+3)}}{C_n^{(n+1)}C_{n+3}^{(n+3)}} + \frac{C_n^{(n+2)}C_{n-1}^{(n)}}{C_n^{(n)}C_{n+2}^{(n+2)}} - \frac{C_n^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}} + \frac{C_n^{(n+3)}}{C_{n+3}^{(n+3)}} - \frac{C_{n-3}^{(n)}}{C_n^{(n)}}\right) P_{0,2}\\
&-(n+1)P_{1,0} + \left(\frac{(n+1)C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{nC_{n-1}^{(n)}}{C_n^{(n)}}\right) P_{1,1} +\\
&\left(\frac{nC_n^{(n+1)}C_{n-1}^{(n)}}{C_n^{(n)}C_{n+1}^{(n+1)}} - \frac{(n+1)C_n^{(n+1)}C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}} + \frac{(n+1)C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \frac{(n-1)C_{n-2}^{(n)}}{C_n^{(n)}}\right) P_{1,2}\\
&+\left(\frac{(n-1)C_n^{(n+1)}C_{n-2}^{(n)}}{C_n^{(n)}C_{n+1}^{(n+1)}} - \frac{nC_{n-1}^{(n)}C_n^{(n+1)}C_{n+1}^{(n+2)}}{C_n^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}} + \frac{(n+1)C_n^{(n+1)}C_{n+1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}\right.\\
&-\frac{(n+1)C_n^{(n+1)}C_{n+1}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+3}^{(n+3)}} + \frac{nC_{n-1}^{(n)}C_n^{(n+2)}}{C_n^{(n)}C_{n+2}^{(n+2)}} - \frac{(n+1)C_n^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}} + \frac{(n+1)C_n^{(n+3)}}{C_{n+3}^{(n+3)}}\\
&\left.-\frac{(n-2)C_{n-3}^{(n)}}{C_n^{(n)}}\right) P_{1,3}
\end{aligned} \tag{64}$$

Case $m = 1,\ s = 3$

$$\begin{aligned}
Ly(x) \equiv &\left(P_{1,0} + P_{1,1}x + P_{1,2}x^2 + P_{1,3}x^3 + P_{1,4}x^4\right) y'(x) + \left(P_{0,0} + P_{0,1}x + P_{0,2}x^2\right.\\
&\left.+P_{0,3}x^3\right) y(x) = \sum_{r=0}^{\sigma} f_r x^r
\end{aligned} \tag{65}$$

Applying the linear differential operator L of (65) on $E_{n+1}(x)$, simplify and equate the corresponding coefficient of $x^{n+i},\ i = 0(1)4$, then solve the resulting equations, we obtain:

$$\phi_n = \frac{\tau_1 C_n^{(n)}}{\gamma_{1,3}} \tag{66}$$

where

$$
\begin{aligned}
\gamma_{1,3} = {} & \Bigg(\frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}}{C_n^{(n)}}\Bigg) P_{0,0} + \Bigg(\frac{C_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \frac{C_{n-2}^{(n)}}{C_n^{(n)}}\Bigg) P_{0,1} \\
& + \Bigg(\frac{C_n^{(n+1)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)} C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} \\
& - \frac{C_n^{(n+1)} C_{n+1}^{(n+3)}}{C_{n+1}^{(n+1)} C_{n+3}^{(n+3)}} + \frac{C_n^{(n+2)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+2}^{(n+2)}} - \frac{C_n^{(n+2)} C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} + \frac{C_n^{(n+3)}}{C_{n+3}^{(n+3)}} - \frac{C_{n-3}^{(n)}}{C_n^{(n)}}\Bigg) P_{0,2} \\
& + \Bigg(\frac{C_n^{(n+1)} C_{n-3}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{C_{n-1}^{(n)} C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} \\
& - \frac{C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} + \frac{C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+4}^{(n+4)}} - \frac{C_n^{(n+1)} C_{n-1}^{(n)} C_{n+1}^{(n+3)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+3}^{(n+3)}} \\
& + \frac{C_n^{(n+1)} C_{n+1}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} - \frac{C_n^{(n+1)} C_{n+1}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+4}^{(n+4)}} + \frac{C_n^{(n+2)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+2}^{(n+2)}} - \frac{C_n^{(n+2)} C_{n+2}^{(n+3)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} \\
& + \frac{C_n^{(n+2)} C_{n+2}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+2}^{(n+2)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} - \frac{C_n^{(n+2)} C_{n+2}^{(n+4)}}{C_{n+2}^{(n+2)} C_{n+4}^{(n+4)}} + \frac{C_n^{(n+3)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+3}^{(n+3)}} - \frac{C_n^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} \\
& + \frac{C_n^{(n+4)}}{C_{n+4}^{(n+4)}} - \frac{C_{n-4}^{(n)}}{C_n^{(n)}}\Bigg) P_{0,3} - (n+1) P_{1,0} + \Bigg(\frac{(n+1) C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{n C_{n-1}^{(n)}}{C_n^{(n)}}\Bigg) P_{1,1} \\
& + \Bigg(\frac{n C_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{(n+1) C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{(n+1) C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \frac{(n-1) C_{n-2}^{(n)}}{C_n^{(n)}}\Bigg) P_{1,2} \\
& + \Bigg(\frac{(n-1) C_n^{(n+1)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{n C_{n-1}^{(n)} C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{(n+1) C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} \\
& - \frac{(n+1) C_n^{(n+1)} C_{n+1}^{(n+3)}}{C_{n+1}^{(n+1)} C_{n+3}^{(n+3)}} + \frac{n C_{n-1}^{(n)} C_n^{(n+2)}}{C_n^{(n)} C_{n+2}^{(n+2)}} - \frac{(n+1) C_n^{(n+2)} C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} + \frac{(n+1) C_n^{(n+3)}}{C_{n+3}^{(n+3)}} \\
& - \frac{(n-2) C_{n-3}^{(n)}}{C_n^{(n)}}\Bigg) P_{1,3} + \Bigg(\frac{(n-2) C_{n-3}^{(n)} C_n^{(n+1)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{(n-1) C_{n+1}^{(n+2)} C_n^{(n+1)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} \\
& + \frac{n C_{n-1}^{(n)} C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} - \frac{(n+1) C_n^{(n+1)} C_{n+1}^{(n+2)} C_{n+2}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} \\
& + \frac{(n+1) C_{n+1}^{(n+2)} C_{n+2}^{(n+4)} C_n^{(n+1)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)} C_{n+4}^{(n+4)}} - \frac{n C_n^{(n+1)} C_{n+1}^{(n+3)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)} C_{n+3}^{(n+3)}} \\
& + \frac{(n+1) C_n^{(n+1)} C_{n+1}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+1}^{(n+1)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} - \frac{(n+1) C_{n+1}^{(n+4)} C_n^{(n+1)}}{C_{n+1}^{(n+1)} C_{n+4}^{(n+4)}} \\
& + \frac{(n-1) C_n^{(n+2)} C_{n-2}^{(n)}}{C_n^{(n)} C_{n+2}^{(n+2)}} - \frac{n C_{n+2}^{(n+3)} C_n^{(n+2)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+2}^{(n+2)} C_{n+3}^{(n+3)}} \\
& + \frac{(n+1) C_n^{(n+2)} C_{n+2}^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+2}^{(n+2)} C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} - \frac{(n+1) C_n^{(n+2)} C_{n+2}^{(n+4)}}{C_{n+2}^{(n+2)} C_{n+4}^{(n+4)}} + \frac{n C_{n-1}^{(n)} C_n^{(n+3)}}{C_n^{(n)} C_{n+3}^{(n+3)}} \\
& - \frac{(n+1) C_n^{(n+3)} C_{n+3}^{(n+4)}}{C_{n+3}^{(n+3)} C_{n+4}^{(n+4)}} + \frac{(n+1) C_n^{(n+4)}}{C_{n+4}^{(n+4)}} - \frac{(n-3) C_{n-4}^{(n)}}{C_n^{(n)}}\Bigg) P_{1,4}
\end{aligned}
\tag{67}
$$

Case $m = 2,\ s = 1$

$$\begin{aligned} Ly(x) &\equiv \left(P_{2,0}+P_{2,1}x+P_{2,2}x^2+P_{2,3}x^3\right)y''(x)+\left(P_{1,0}+P_{1,1}x+P_{1,2}x^2\right)y'(x) \\ &\quad+\left(P_{0,0}+P_{0,1}x\right)y(x)=\sum_{r=0}^{\sigma} f_r x^r \end{aligned} \tag{68}$$

The linear differential operator L of equation (68) is:

$$\begin{aligned} L &= \left(P_{2,0}+P_{2,1}x+P_{2,2}x^2+P_{2,3}x^3\right)\frac{d^2}{dx^2}+\left(P_{1,0}+P_{1,1}x+P_{1,2}x^2\right)\frac{d}{dx} \\ &\quad+\left(P_{0,0}+P_{0,1}x\right) \end{aligned} \tag{69}$$

Applying linear differential operator L (that is equation 69) on (43), simplify and equate the corresponding coefficient of $x^{n+i-1},\ i = 0(1)3$, then solve the resulting equations, we obtain:

$$\phi_n = \frac{\tau_1 C_{n-1}^{(n-1)}}{\gamma_{2,1}} \tag{70}$$

where

$$\begin{aligned} \gamma_{2,1} = &\left(\frac{C_{n-2}^{(n-1)}C_{n-1}^{(n)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}}-\frac{C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{0,0}+\left(\frac{C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}\right. \\ &-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+2)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}}+\frac{C_{n-1}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}} \\ &\left.-\frac{C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}+\frac{C_{n-1}^{(n+2)}}{C_{n+2}^{(n+2)}}-\frac{C_{n-4}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{0,1}+\left(\frac{(n+1)C_{n-1}^{(n)}}{C_{n}^{(n)}}-\frac{nC_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{1,0} \\ &+\left(\frac{nC_{n-1}^{(n)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{(n+1)C_{n-1}^{(n)}C_{n}^{(n+1)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{(n+1)C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}}-\frac{(n-1)C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{1,1} \\ &+\left(\frac{(n-1)C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{nC_{n-1}^{(n)}C_{n}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{(n+1)C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}\right. \\ &-\frac{(n+1)C_{n-1}^{(n)}C_{n}^{(n+2)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}}+\frac{nC_{n-1}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}-\frac{(n+1)C_{n+1}^{(n+2)}C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}+\frac{(n+1)C_{n-1}^{(n+2)}}{C_{n+2}^{(n+2)}} \\ &\left.-\frac{(n-2)C_{n-4}^{(n-1)}}{C_{n-1}^{(n-1)}}+\right)P_{1,2}-n(n+1)P_{2,0}+\left(\frac{n(n+1)C_{n-1}^{(n)}}{C_{n}^{(n)}}-\frac{(n-1)nC_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{2,1} \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{(n-1)nC_{n-1}^{(n)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{n(n+1)C_{n-1}^{(n)}C_{n}^{(n+1)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{n(n+1)C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}}\right.\\
&\left.-\frac{(n-1)(n-2)C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{2,2}+\left(\frac{(n-1)(n-2)C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}\right.\\
&-\frac{n(n-1)C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{n(n+1)C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}\\
&-\frac{n(n+1)C_{n-1}^{(n)}C_{n}^{(n+2)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}}+\frac{n(n-1)C_{n-1}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}\\
&\left.-\frac{n(n+1)C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}+\frac{n(n+1)C_{n-1}^{(n+2)}}{C_{n+2}^{(n+2)}}-\frac{(n-2)(n-3)C_{n-4}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{2,3}
\end{aligned}
\tag{71}
$$

Case $m=2,\ s=2$

$$
\begin{aligned}
Ly(x)\equiv&\left(P_{2,0}+P_{2,1}x+P_{2,2}x^2+P_{2,3}x^3+P_{2,4}x^4\right)y''(x)+\left(P_{1,0}+P_{1,1}x+P_{1,2}x^2\right.\\
&\left.+P_{1,3}x^3\right)y'(x)+\left(P_{0,0}+P_{0,1}x+P_{0,2}x^2\right)y(x)=\sum_{r=0}^{\sigma}f_rx^r
\end{aligned}
\tag{72}
$$

The linear differential operator L of equation (72) is:

$$
\begin{aligned}
L=&\left(P_{2,0}+P_{2,1}x+P_{2,2}x^2+P_{2,3}x^3+P_{2,4}x^4\right)\frac{d^2}{dx^2}+\left(P_{1,0}+P_{1,1}x+P_{1,2}x^2\right.\\
&\left.+P_{1,3}x^3\right)\frac{d}{dx}+\left(P_{0,0}+P_{0,1}x+P_{0,2}x^2\right)
\end{aligned}
\tag{73}
$$

Applying linear differential operator L (that is equation 73) on (43), simplify and equate the corresponding coefficient of x^{n+i-1}, $i=0(1)3$, then solve the resulting equations, we obtain:

$$
\phi_n=\frac{\tau_1C_{n-1}^{(n-1)}}{\gamma_{2,2}} \tag{74}
$$

where

$$
\begin{aligned}
\gamma_{2,2}=&\left(\frac{C_{n-2}^{(n-1)}C_{n-1}^{(n)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}}-\frac{C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{0,0}+\left(\frac{C_{n-1}^{(n)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}\right.\\
&-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+2)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}}+\frac{C_{n-1}^{(n+1)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}\\
&+\left(\frac{C_{n-1}^{(n)}C_{n-4}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}}+\frac{C_{n-2}^{(n-1)}C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}\right.\\
&-\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}+\frac{C_{n-1}^{(n)}C_{n}^{(n+1)}C_{n+1}^{(n+3)}}{C_{n}^{(n)}C_{n+1}^{(n+1)}C_{n+3}^{(n+3)}}-\frac{C_{n-1}^{(n)}C_{n-2}^{(n-1)}C_{n}^{(n+2)}}{C_{n-1}^{(n-1)}C_{n}^{(n)}C_{n+2}^{(n+2)}}\\
&+\frac{C_{n-1}^{(n)}C_{n}^{(n+2)}C_{n-2}^{(n+3)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}-\frac{C_{n-1}^{(n)}C_{n}^{(n+3)}}{C_{n}^{(n)}C_{n+3}^{(n+3)}}+\frac{C_{n-1}^{(n+1)}C_{n-3}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}-\frac{C_{n-2}^{(n-1)}C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}\\
&+\frac{C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}-\frac{C_{n-1}^{(n+1)}C_{n+1}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+3}^{(n+3)}}+\frac{C_{n-1}^{(n+2)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+2}^{(n+2)}}-\frac{C_{n-1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C^{(n+3)}_{n-1}}{C^{(n+3)}_{n+3}}-\frac{C^{(n-1)}_{n-5}}{C^{(n-1)}_{n-1}}\Bigg)P_{0,2}+\Bigg(\frac{(n+1)C^{(n)}_{n-1}}{C^{(n)}_{n}}-\frac{nC^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}}\Bigg)P_{1,0}+\Bigg(\frac{nC^{(n)}_{n-1}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}\\
&-\frac{(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}}{C^{(n)}_{n}C^{(n+1)}_{n+1}}+\frac{(n+1)C^{(n+1)}_{n-1}}{C^{(n+1)}_{n+1}}-\frac{(n-1)C^{(n-1)}_{n-3}}{C^{(n-1)}_{n-1}}\Bigg)P_{1,1}+\Bigg(\frac{(n-1)C^{(n)}_{n-1}C^{(n-1)}_{n-3}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}\\
&-\frac{nC^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}}+\frac{(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}-\frac{(n+1)C^{(n)}_{n-1}C^{(n+2)}_{n}}{C^{(n)}_{n}C^{(n+2)}_{n+2}}\\
&+\frac{nC^{(n+1)}_{n-1}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n+1)}_{n+1}}-\frac{(n+1)C^{(n+2)}_{n+1}C^{(n+1)}_{n-1}}{C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}+\frac{(n+1)C^{(n+2)}_{n-1}}{C^{(n+2)}_{n+2}}-\frac{(n-2)C^{(n-1)}_{n-4}}{C^{(n-1)}_{n-1}}+\Bigg)P_{1,2}\\
&+\Bigg(\frac{(n-2)C^{(n)}_{n-1}C^{(n-1)}_{n-4}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}-\frac{(n-1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n-1)}_{n-3}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}}+\frac{nC^{(n-1)}_{n-2}C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}\\
&-\frac{(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}C^{(n+3)}_{n+2}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}C^{(n+3)}_{n+3}}+\frac{(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+3)}_{n+1}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+3)}_{n+3}}-\frac{nC^{(n-1)}_{n-2}C^{(n)}_{n-1}C^{(n+2)}_{n}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+2)}_{n+2}}\\
&+\frac{(n+1)C^{(n)}_{n-1}C^{(n+2)}_{n}C^{(n+3)}_{n+2}}{C^{(n)}_{n}C^{(n+2)}_{n+2}C^{(n+3)}_{n+3}}-\frac{(n+1)C^{(n)}_{n-1}C^{(n+3)}_{n}}{C^{(n)}_{n}C^{(n+3)}_{n+3}}+\frac{(n-1)C^{(n-1)}_{n-3}C^{(n+1)}_{n-1}}{C^{(n-1)}_{n-1}C^{(n+1)}_{n+1}}\\
&-\frac{nC^{(n-1)}_{n-2}C^{(n+1)}_{n-1}C^{(n+2)}_{n+1}}{C^{(n-1)}_{n-1}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}+\frac{(n+1)C^{(n+1)}_{n-1}C^{(n+2)}_{n+1}C^{(n+3)}_{n+2}}{C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}C^{(n+3)}_{n+3}}-\frac{(n+1)C^{(n+1)}_{n-1}C^{(n+3)}_{n+1}}{C^{(n+1)}_{n+1}C^{(n+3)}_{n+3}}\\
&+\frac{nC^{(n-1)}_{n-2}C^{(n+2)}_{n-1}}{C^{(n-1)}_{n-1}C^{(n+2)}_{n+2}}-\frac{(n+1)C^{(n+2)}_{n-1}C^{(n+3)}_{n+2}}{C^{(n+2)}_{n+2}C^{(n+3)}_{n+3}}+\frac{(n+1)C^{(n+3)}_{n-1}}{C^{(n+3)}_{n+3}}-\frac{(n-3)C^{(n-1)}_{n-5}}{C^{(n-1)}_{n-1}}\Bigg)P_{1,3}\\
&-n(n+1)P_{2,0}+\Bigg(\frac{n(n+1)C^{(n)}_{n-1}}{C^{(n)}_{n}}-\frac{(n-1)nC^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}}\Bigg)P_{2,1}+\Bigg(\frac{(n-1)nC^{(n)}_{n-1}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}\\
&-\frac{n(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}}{C^{(n)}_{n}C^{(n+1)}_{n+1}}+\frac{n(n+1)C^{(n+1)}_{n-1}}{C^{(n+1)}_{n+1}}-\frac{(n-1)(n-2)C^{(n-1)}_{n-3}}{C^{(n-1)}_{n-1}}\Bigg)P_{2,2}\\
&+\Bigg(\frac{(n-1)(n-2)C^{(n)}_{n-1}C^{(n-1)}_{n-3}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}-\frac{n(n-1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}}\\
&+\frac{n(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}-\frac{n(n+1)C^{(n)}_{n-1}C^{(n+2)}_{n}}{C^{(n)}_{n}C^{(n+2)}_{n+2}}+\frac{n(n-1)C^{(n+1)}_{n-1}C^{(n-1)}_{n-2}}{C^{(n-1)}_{n-1}C^{(n+1)}_{n+1}}\\
&-\frac{n(n+1)C^{(n+1)}_{n-1}C^{(n+2)}_{n+1}}{C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}+\frac{n(n+1)C^{(n+2)}_{n-1}}{C^{(n+2)}_{n+2}}-\frac{(n-2)(n-3)C^{(n-1)}_{n-4}}{C^{(n-1)}_{n-1}}\Bigg)P_{2,3}\\
&+\Bigg(\frac{(n-2)(n-3)C^{(n-1)}_{n-4}C^{(n)}_{n-1}}{C^{(n-1)}_{n-1}C^{(n)}_{n}}-\frac{(n-1)(n-2)C^{(n-1)}_{n-3}C^{(n)}_{n-1}C^{(n+1)}_{n}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}}\\
&+\frac{n(n-1)C^{(n-1)}_{n-2}C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}}-\frac{n(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+2)}_{n+1}C^{(n+3)}_{n+2}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+2)}_{n+2}C^{(n+3)}_{n+3}}\\
&+\frac{n(n+1)C^{(n)}_{n-1}C^{(n+1)}_{n}C^{(n+3)}_{n+1}}{C^{(n)}_{n}C^{(n+1)}_{n+1}C^{(n+3)}_{n+3}}-\frac{n(n-1)C^{(n)}_{n-1}C^{(n-1)}_{n-2}C^{(n+2)}_{n}}{C^{(n-1)}_{n-1}C^{(n)}_{n}C^{(n+3)}_{n+3}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n(n+1)C_{n-1}^{(n)}C_{n}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n}^{(n)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}-\frac{n(n+1)C_{n-1}^{(n)}C_{n}^{(n+3)}}{C_{n}^{(n)}C_{n+3}^{(n+3)}}+\frac{(n-2)(n-1)C_{n-3}^{(n-1)}C_{n-1}^{(n+1)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}}\\
&-\frac{n(n-1)C_{n-2}^{(n-1)}C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}}{C_{n-1}^{(n-1)}C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}}+\frac{n(n+1)C_{n-1}^{(n+1)}C_{n+1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}\\
&-\frac{n(n+1)C_{n-1}^{(n+1)}C_{n+1}^{(n+3)}}{C_{n+1}^{(n+1)}C_{n+3}^{(n+3)}}+\frac{n(n-1)C_{n-1}^{(n+2)}C_{n-2}^{(n-1)}}{C_{n-1}^{(n-1)}C_{n+2}^{(n+2)}}\\
&\left.-\frac{n(n+1)C_{n-1}^{(n+2)}C_{n+2}^{(n+3)}}{C_{n+2}^{(n+2)}C_{n+3}^{(n+3)}}+\frac{n(n+1)C_{n-1}^{(n+3)}}{C_{n+3}^{(n+3)}}-\frac{(n-3)(n-4)C_{n-5}^{(n-1)}}{C_{n-1}^{(n-1)}}\right)P_{2,4}
\end{aligned}
\tag{75}
$$

Observing the trend of equations (54), (64), (67), (71) and equation (75), we derived the general formula for $\gamma_{m,s}$ as:

$$
\begin{aligned}
\gamma_{m,s}=&\sum_{r=0}^{m}\sum_{k=0}^{r}\sum_{i=0}^{m-r+k} r!\binom{n-i+1}{r}P_{r,k}\frac{C_{n-m-i+1}^{(n-m+1)}}{C_{n-m+1}^{(n-m+1)}}\beta_{m-r+k-i}\\
&+\sum_{k=0}^{m}\sum_{i=1}^{s}\sum_{r=0}^{m+i} k!\binom{n-r+1}{k}P_{k,k+i}\frac{C_{n-m-r+1}^{(n-m+1)}}{C_{n-m+1}^{(n-m+1)}}\beta_{m+i-r}
\end{aligned}
\tag{76}
$$

where

$$\beta_i=\sum_{r=1}^{i}-\frac{C_{n-m+r}^{(n-m+i+1)}}{C_{n-m+i+1}^{(n-m+i+1)}}\beta_{r-1}\ \ i=1,2,\ldots,(m+s),\ \ \beta_0=-1, \tag{77}$$

$$\phi_n=\frac{\tau_1 C_{n-m+1}^{(n-m+1)}}{\gamma_{m,s}} \tag{78}$$

and

$$\epsilon_n^*=\max\left|(e_n)_{(n+1)}(x):\ a\le x\le b\right|=\left|\frac{\tau_1}{\gamma_{m,s}}\right| \tag{79}$$

is the error estimate for the equation (1) and we obtained the value of τ_1 from equation (24).

In the process of solving numerical problems, we assume $C_{-r}^{(n)}=0$. The number of $P_{i,j}$'s in equation (76) correspond to the expression $\frac{1}{2}(m+1)(2+m+2s)$ and the number of terms in the coefficient of each $P_{i,j}$'s correspond to the expression 2^{j+m-i}.

Mathematica Code for $\gamma_{m,s}$:

$$
\begin{aligned}
\gamma_{m,s}=&\sum_{r=0}^{m}\sum_{k=0}^{r}\sum_{i=0}^{m-r+k} r!\mathrm{Binomial}[n-i+1,r]P_{r,k}\frac{\overset{n-m+1}{\underset{n-m-i+1}{C}}}{\overset{n-m+1}{\underset{n-m+1}{C}}}G_{m-r+k-i}\\
&+\sum_{k=0}^{m}\sum_{i=1}^{s}\sum_{r=0}^{m+i} k!\mathrm{Binomial}[n-r+1,k]P_{k,k+i}\frac{\overset{n-m+1}{\underset{n-m-r+1}{C}}}{\overset{n-m+1}{\underset{n-m+1}{C}}}G_{m+i-r}
\end{aligned}
$$

We validate equation (76) in theorem 1 below.

Theorem 1:
The parametre $\gamma_{m,s}$ in the general error term ξ_n of the error:

$$\xi_n = \max\{|\ y(x) - y_n(x)\ |: a \leq x \leq b\} \tag{80}$$

for the m-th order differential equation:

$$Ly(x) \equiv \sum_{r=0}^{m} \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) y^r(x) = \sum_{r=0}^{\sigma} f_r x^r, a \leq x \leq b \tag{81}$$

is given by

$$\begin{aligned}\gamma_{m,s} =& \sum_{r=0}^{m}\sum_{k=0}^{r}\sum_{i=0}^{m-r+k} r! \left(\begin{array}{c} n-i+1 \\ r \end{array} \right) P_{r,k} \frac{C_{n-m-i+1}^{(n-m+1)}}{C_{n-m+1}^{(n-m+1)}} \beta_{m-r+k-i} \\ &+ \sum_{k=0}^{m}\sum_{i=1}^{s}\sum_{r=0}^{m+i} k! \left(\begin{array}{c} n-r+1 \\ k \end{array} \right) P_{k,k+i} \frac{C_{n-m-r+1}^{(n-m+1)}}{C_{n-m+1}^{(n-m+1)}} \beta_{m+i-r}.\end{aligned} \tag{82}$$

where

$$\beta_i = \sum_{r=1}^{i} - \frac{C_{n-m+r}^{(n-m+i+1)}}{C_{n-m+i+1}^{(n-m+i+1)}} \beta_{r-1} \ i = 1, 2, \ldots, m + s \text{and } \beta_0 = -1. \tag{83}$$

where s is the overdetermination number given by equation (8), n is the order of the error term and $y_n(x)$ is the Tau approximant given by equation (22)of the solution $y(x)$.

Proof
We shall employ the principle of mathematical induction to validate $\gamma_{m,s}$ and this shall be accomplish by varying m and fixed s
For $m = 1, \ s = 1$

$$\begin{aligned}\gamma_{1,1} =& \sum_{r=0}^{1}\sum_{k=0}^{r}\sum_{i=0}^{1-r+k} r! \left(\begin{array}{c} n-i+1 \\ r \end{array} \right) P_{r,k} \frac{C_{n-i}^{(n)}}{C_n^{(n)}} \beta_{1-r+k-i} \\ &+ \sum_{k=0}^{1}\sum_{i=1}^{1}\sum_{r=0}^{1+i} k! \left(\begin{array}{c} n-r+1 \\ k \end{array} \right) P_{k,k+i} \frac{C_{n-r}^{(n)}}{C_n^{(n)}} \beta_{1+i-r}\end{aligned}$$

After simplification we obtain:

$$\begin{aligned}\gamma_{1,1} =& \left(\frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}}{C_n^{(n)}} \right) P_{0,0} + \left(\frac{C_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} - \frac{C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \frac{C_{n-2}^{(n)}}{C_n^{(n)}} \right) P_{0,1} \\ &- (n+1)P_{1,0} + \left(\frac{(n+1)C_n^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{nC_{n-1}^{(n)}}{C_n^{(n)}} \right) P_{1,1} + \left(\frac{nC_n^{(n+1)} C_{n-1}^{(n)}}{C_n^{(n)} C_{n+1}^{(n+1)}} \right. \\ &\left. - \frac{(n+1)C_n^{(n+1)} C_{n+1}^{(n+2)}}{C_{n+1}^{(n+1)} C_{n+2}^{(n+2)}} + \frac{(n+1)C_n^{(n+2)}}{C_{n+2}^{(n+2)}} - \frac{(n-1)C_{n-2}^{(n)}}{C_n^{(n)}} \right) P_{1,2}\end{aligned}$$

This conform with equation(54), hence equation (25) is true for $m = 1, \ s = 1$.

Assume $m = v,\ s = 1$

$$\gamma_{v,1} = \sum_{r=0}^{v} \sum_{k=0}^{r} \sum_{i=0}^{v-r+k} r! \begin{pmatrix} n-i+1 \\ r \end{pmatrix} P_{r,k} \frac{C_{n-v-i+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v-r+k-i} + \sum_{k=0}^{v} \sum_{i=1}^{1} \sum_{r=0}^{v+i} k! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v+i-r}$$

For $m = v+1,\ s = 1$

$$\begin{aligned}\gamma_{v+1,1} = &\sum_{r=0}^{v} \sum_{k=0}^{r} \sum_{i=0}^{v-r+k} r! \begin{pmatrix} n-i+1 \\ r \end{pmatrix} P_{r,k} \frac{C_{n-v-i+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v-r+k-i} \\ &+ \sum_{k=0}^{v} \sum_{i=1}^{1} \sum_{r=0}^{v+i} k! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v+i-r} \\ &+ (v+1)! \begin{pmatrix} n-v \\ v+1 \end{pmatrix} P_{v+1,v+1} \frac{C_{n-v-i}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-i+1} \\ &+ (v+1)! \begin{pmatrix} n-r+1 \\ v+1 \end{pmatrix} P_{v+1,v+2} \frac{C_{n-v-r}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+2}\end{aligned}$$

After simplification, we obtain:

$$\gamma_{v+1,1} = \sum_{r=0}^{v+1} \sum_{k=0}^{r} \sum_{i=0}^{v-r+k+1} r! \begin{pmatrix} n-i+1 \\ r \end{pmatrix} P_{r,k} \frac{C_{n-v-i}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+k-i+1} + \sum_{k=0}^{v+1} \sum_{r=0}^{v+2} k! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+1} \frac{C_{n-v-r}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+1}$$

Therefore $\gamma_{m,s}$ is true for $m = v+1$ and $s = 1$, we assume $\gamma_{m,s}$ is true for $m = v+1$ and $s = t$

And lastly for $s = t+1$ and $m = v+1$, we have:

$$\begin{aligned}\gamma_{v+1,t+1} = &\sum_{r=0}^{v} \sum_{k=0}^{r} \sum_{i=0}^{v-r+k} r! \begin{pmatrix} n-i+1 \\ r \end{pmatrix} P_{r,k} \frac{C_{n-v-i+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v-r+k-i} \\ &+ \sum_{k=0}^{v} \sum_{i=1}^{1} \sum_{r=0}^{v+i} k! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r+1}^{(n-v+1)}}{C_{n-v+1}^{(n-v+1)}} \beta_{v+i-r} \\ &+ (v+1)! \begin{pmatrix} n-v \\ v+1 \end{pmatrix} P_{v+1,v+1} \frac{C_{n-v-i}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-i+1} \\ &+ (v+1)! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+t+1}\end{aligned} \tag{84}$$

Simplifying equation (84), we have:

$$\begin{aligned}\gamma_{v+1,t+1} =& \gamma_{m,t} + (v+1)! \begin{pmatrix} n-v \\ v+1 \end{pmatrix} P_{v+1,v+1} \frac{C_{n-v-i}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-i+1} \\ &+ (v+1)! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+t+1}\end{aligned} \tag{85}$$

Equation (85) becomes:

$$\begin{aligned}\gamma_{v+1,t+1} =& \sum_{r=0}^{v} \sum_{k=0}^{r} \sum_{i=0}^{v-r+k} r! \begin{pmatrix} n-i+1 \\ r \end{pmatrix} P_{r,k} \frac{C_{n-v-i}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v-r+k-i+1} \\ &+ \sum_{k=0}^{v+1} \sum_{i=1}^{t+1} \sum_{r=0}^{v+i+1} k! \begin{pmatrix} n-r+1 \\ k \end{pmatrix} P_{k,k+i} \frac{C_{n-v-r}^{(n-v)}}{C_{n-v}^{(n-v)}} \beta_{v+i-r+1}\end{aligned}$$

Therefore $\gamma_{m,s}$ is true for all $m,$ and s

4. NUMERICAL EXPERIMENTS

The results reported in previous section are employed here to illustrate our results, we illustrate the experiments by solving numerical problems from literature review. The exact errors are defined as $\varepsilon_n = \max\{\mid y(x) - y_n(x) \mid : a \leq x \leq b\}$, where $\{x_k\} = (0.01k), k = 0(1)100$

Experiment 4.1[16]

$$Ly(x) = y'(x) + 2xy(x) = 4x$$
$$\text{subject to initial condition}\, y(0) = 3$$

with analytical solution $y(x) = 2 + \exp(-x^2)$

Experiment 4.2[16]

$$Ly(x) = y'(x) - x^2y(x) = 0$$
$$\text{subject to initial condition}\, y(0) = 1$$

with analytical solution $y(x) = \exp(\frac{1}{3}x^3)$

Experiment 4.3[3]

$$Ly(x) = y''(x) - 2(1+2x^2)y(x) = 0$$
$$\text{subject to initial condition}\, y(0) = 1,\; y'(0) = 0$$

with analytical solution $y(x) = \exp(x^2)$

Experiment 4.4[28]

$$Ly(x) = 2(1+x)y'(x) + y(x) = 0,\; 0 \leq x \leq 1$$
$$\text{subject to initial condition}\, y(0) = 1$$

with analytical solution $y(x) = (1+x)^{\frac{-1}{2}}$

Experiment 4.5[28]

$$y''(x) - (1-x)y'(x) + y(x) = 0,\ 0 \le x \le 1$$
$$\text{subject to initial condition}\, y(0) = 1, y'(0) = 1$$

with analytical solution $\exp(x - \frac{x^2}{2})$

Experiment 4.6[3]

$$y^{iv}(x) - 3601y^{ii}(x) + 3600y(x) = -1 + 1800x^2,\ 0 \le x \le 1$$
$$\text{subject to initial condition}\, y(0) = y^{i}(0) = y''(0) = y^{iii}(0) = 1$$

with analytical solution:
$\frac{1}{2}\exp(-x)\left(-1 + 2\exp(x) + \exp(2x) + x^2\exp(x)\right)$

Experiment 4.7[26]

$$\cos^2(x)\frac{d^2y(x)}{dx^2} + \frac{1}{2}\sin(2x)\frac{dy}{dx} + \frac{3}{4}y(x) = 0,\ 0 \le x \le 1$$
$$\text{subject to initial condition}\, y(0) = 1\ , y'(0) = 1$$

with analytical solution $2\sqrt{\cos(x)}\sin\left(\frac{x}{2}\right) + \sqrt{\cos(x)}\cos\left(\frac{x}{2}\right)$

Table 4.1:Exact Errors for Experiment 4.1-4.7

Experiment	Degree 5	Degree 6	Degree 7
Experiment 4.1	3.57×10^{-5}	1.97×10^{-5}	1.10×10^{-6}
Experiment 4.2	2.07×10^{-4}	2.96×10^{-5}	4.50×10^{-6}
Experiment 4.3	1.06×10^{-2}	1.23×10^{-3}	1.40×10^{-4}
Experiment 4.4	3.08×10^{-5}	4.24×10^{-6}	6.64×10^{-7}
Experiment 4.5	9.42×10^{-5}	2.62×10^{-5}	1.81×10^{-7}
Experiment 4.6	1.70×10^{-5}	1.60×10^{-6}	2.48×10^{-9}
Experiment 4.7	2.77×10^{-4}	9.89×10^{-5}	6.54×10^{-5}

Table 4.2:Estimate Error for Experiment 4.1-4.7

Experiment	Degree 5	Degree 6	Degree 7
Experiment 4.1	4.08×10^{-5}	2.77×10^{-5}	1.56×10^{-6}
Experiment 4.2	3.28×10^{-4}	4.43×10^{-5}	7.03×10^{-6}
Experiment 4.3	6.34×10^{-3}	5.76×10^{-4}	1.18×10^{-4}
Experiment 4.4	2.90×10^{-5}	4.60×10^{-6}	7.37×10^{-7}
Experiment 4.5	7.08×10^{-5}	2.05×10^{-5}	2.32×10^{-7}
Experiment 4.6	7.68×10^{-6}	5.69×10^{-7}	5.83×10^{-9}
Experiment 4.7	4.33×10^{-4}	7.94×10^{-5}	3.44×10^{-5}

The interesting fact that can be observed in the results presented above is the correlation between the exact error and estimates error, which so that we can compute the estimates error without actually compute the approximate solution of the problem. These further emphasized the desirability of the reported formula.

5. CONCLUSION

We have obtained the generalized error estimation for the recursive Tau method using chebyshev polynomials as the perturbation terms and applied the results on some numerical problems. Our results are compared with some existing ones in the literature and we found that they are in good agreement. We also obtained the recursive approximate to non-linear ordinary differential equations and compute the exact error ($\xi_{n,k}$). We observed that increase in degree of approximation reduce the exact error and estimates error of the problem. Finally, because of the accuracy, effectiveness, and simplicity of the method presented in this work, we recommend its application in finding the approximate solution and estimates error of an ODEs.

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DEPARTMENT OF STATISTICS AND MATHEMATICAL SCIENCES, KWARA STATE UNIVERSITY, MALETE, NIGERIA

E-mail address: issakazeem@yahoo.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA

E-mail addresses: raphade@unilorin.edu.ng

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA

E-mail addresses: yisa.bm@unilorin.edu.ng