Journal of the Nigerian Mathematical Society
Vol. 43, Issue 1, pp. 81 - 88, 2024

ASYMPTOTIC STABILITY AND BOUNDEDNESS CRITERIA FOR A CERTAIN SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we examined some criteria for the stability and boundedness of solutions to certain second order nonlinear differential equation

$$x'' + b(t)f(x,x') + c(t)g(x)h(x') = p(t,x,x'),$$

where b, c, f, g, h and p are real valued functions which depend on the argument displayed explicitly. By applying a suitable Lyapunov function to study the qualitative properties mentioned earlier, we are able to extablish the asymptotic stability and boundedness of solutions. Examples on the stability and boundednress of solutions are hereby included to corroborate our results.

1. INTRODUCTION

Boundedness and stability problems of generalized Liénard equations have been extensively studied over the years. As a survey, one can refer to the book by Sansone and Conti [15] and the papers by Adams et al. [2], [3], Adeyanju et al. [4], Athanassov [1], Bhatia [6], Bihari [7], Burton and Grimmer [8], Chang [9], Graef and Spikes [10], Hatvani [11], Lalli [12], Nápoles Valdés [13], Sugie and Amano [16], Tunç [18], [17] and Wong and Burton [19].

In this paper, we consider the second order nonlinear differential equations of the form

$$x'' + b(t)f(x,x') + c(t)g(x)h(x') = p(t,x,x'), \tag{1}$$

Received by the editors January 25, 2023; Revised: February 05, 2024; Accepted: February 29, 2024
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp. it/jnms/
2010 *Mathematics Subject Classification.* 34C11, 34D20.
Key words and phrases. Stability; boundedness; differential equations; second order; Lyapunov function.

where the functions b,c,f,g,h and p are assumed to be real-valued continuous functions which depend at most on the respective arguments displayed explicitly.

Now, in 1969, Lalli [12] discussed stability and boundedness of solutions of certain differential equation of the form

$$(r(t)u')' + a(t)f(u)g(u') = q(t)$$

in which cases $q(t) = 0$ and $q(t) \neq 0$ were considered.

Zarghamee and Mehri [20], gave results on the stability and boundedness of solutions to a class of second order differential equation of the form

$$(r(t)u')' + a(t)f(u)g(u') + b(t)h(u)m(u') = 0,$$

where h and m are chosen so that $h(u) \geq 0$ and $u'm(u') \geq 0$. Further demonstration on the boundedness property of the solutions of the related nonhomogeneous differential equation:

$$(r(t)u')' + a(t)f(u)g(u') + b(t)h(u)m(u') = q(t)$$

was also considered.

In another contribution, Athanassov [1] considered the boundedness criteria for solutions of the second order nonlinear differential Eq. (1) through the use of "integral inequalities" in [5] and achieved some interesting results.

The motivation for this research comes from the work of Athanassov [1] and that of [12], [18] and [20]. Our goal is therefore to use a suitable Lyapunov function to study the stability and boundedness of solutions to Eq. (1) for the cases in which $p(t,x,x') = 0$ and $p(t,x,x') \neq 0$ and thereby improve on the results established by Athanassov [1].

2. PRELIMINARIES AND NOTATIONS

We will have an occasion to use the following well-known lemma due to Bellman (See [1],[5]) which is also known as Gronwall's inequality.

Lemma 1. If u and v are real valued functions, defined and nonnegative for $t \geq t_0$, $u,v \in L^1(t_0,t)$ and if

$$u(t) \leq c + \int_{t_0}^{t} u(s)v(s)ds,$$

for some positive constant c, then

$$u(t) \leq c\exp\left(\int_{t_0}^{t} v(s)ds\right).$$

Let

$$X' = F(t,X) \tag{2}$$

Theorem 2 [14]. Assume that there exists a function $V(t,X)$ defined for $t \geq 0$, $|X| < \delta$ (δ is a positive constant) continuous with the following properties:
(i) $V(t,0) \equiv 0$,
(ii) $V(t,X) \geq a(|X|)$,
where $a(r)$ is continuous and monotonically increasing, a(0) = 0,
(iii) $V'_{(2)}(t,X) \leq -c(|X|)$,
where $c(r)$ is continuous on $[0,\delta]$ and positive, and if $F(t,X)$ is bounded, the zero solution of Eq. (2) is asymptotically stable.

The following notations are used. We denote the real line $\mathbb{R} = (0,\infty)$, $I = [0,\infty)$ and an absolute value by $|\cdot|$. $L^1(\mathbb{R})$ denotes the set of Lebesque integrable functions on $\mathbb{R}$.

3. MAIN RESULTS

We consider first, the stability result when $p(t,x,x') = 0$.
Let $x' = y$, and transform Eq. (1) into the equivalent system

$$\begin{aligned} x' &= y \\ y' &= -b(t)f(x,y) - c(t)g(x)h(y). \end{aligned} \tag{3}$$

Basic Assumptions

The following are the basic assumptions used to formulate the main results of (1):
(c_1) $b(t) \geq b_0$, where $b \in C(I,I)$;
(c_2) $f \in C(\mathbb{R}^2,\mathbb{R}), g \in C(\mathbb{R},\mathbb{R}), h \in C(\mathbb{R},\mathbb{R}^+)$;
(c_3) $c(t) \geq c_0 > 0,$, where $c \in C(I,\mathbb{R}^+)$;
(c_4) $\lim\limits_{y\to\pm\infty} \int_0^y \frac{\tau}{h(\tau)} d\tau = \infty = \lim\limits_{x\to\pm\infty} \int_0^x g(\tau)d\tau$;
(c_5) $\frac{f(x,y)}{y} \geq \eta \ (x,y) \in \mathbb{R}^2, y \neq 0$;
(c_6) $\frac{g(x)}{x} \geq \alpha \ (x \neq 0)$ and $\int_0^x \frac{g(\tau)}{\tau}\tau d\tau \geq \frac{1}{2}\alpha x^2$ for all $x \in \mathbb{R}$.
(c_7) $\frac{1}{h(y)} \geq \beta \ (h(y) \neq 0)$ and $\int_0^y \frac{\tau}{h(\tau)} d\tau \geq \frac{1}{2}\beta y^2$ for all $y \in \mathbb{R}$.

Theorem 3. If, in addition to $(c_1)-(c_7)$, we assume that:
(c_8) $c'(t) < 0$ and there exist a positive constant ξ and such that $\lim\limits_{t\to\infty} c(t) = \xi$ then the solution of the system (3) is asymptotically stable in the sense of Lyapunov.

Proof. Consider the function

$$V(t,x,y) = c(t)\int_0^x g(\tau)d\tau + \int_0^y \frac{\tau}{h(\tau)}d\tau. \qquad (4)$$

This can be written as

$$V(t,x,y) = c(t)\int_0^x \frac{g(\tau)}{\tau}\tau d\tau + \int_0^y \frac{\tau}{h(\tau)}d\tau.$$

In view of the assumptions c_3, c_6 and c_7 we have that the function

$$\begin{aligned} V(t,x,y) &\geq c_0\frac{1}{2}\alpha x^2 + \frac{1}{2}\beta y^2 \\ &\geq A_1(x^2+y^2), \end{aligned}$$

where $A_1 = \min\left\{c_0\frac{\alpha}{2}, \frac{\beta}{2}\right\}$.

Calculating $V' \equiv V'(t,x,y)$ by differentiating Eq. (4) with respect to t along the system (3) as follows:

$$\begin{aligned} V' &= c'(t)\int_0^x g(\tau)d\tau + c(t)g(x)x' + \frac{y}{h(y)}y' \\ &= c'(t)\int_0^x g(\tau)d\tau + c(t)g(x)y - b(t)f(x,y)\frac{y}{h(y)} - c(t)g(x)y \\ &= c'(t)\int_0^x \frac{g(\tau)}{\tau}\tau d\tau - b(t)\frac{f(x,y)y^2}{h(y)y} \end{aligned}$$

By the assumptions (c_1) to (c_5), we have

$$V' \leq \frac{1}{2}c'(t)\lambda x^2 - b_0\eta\frac{1}{h(y)}y^2.$$

There exists a positive constant A_2 small enough such that

$$V' \leq -A_2(x^2+y^2).$$

Thus, every solution of Eq. (1) is asymptotically stable. This completes the proof of Theorem 3.

The second main result is the following Theorem on the boundedness of solution. For the case in which $p(t,x,y) \neq 0$ in Eq. (1).

Theorem 4. Let us assume that the conditions of Theorem 1 holds. In addition, we have that

$$|p(t,x,y)| \leq e(t)h(y),$$

where $e(t) \in L^1(0,\infty)$ [$L^1(0,\infty)$ is a space of integrable Lebesque function]. Then, there exists a positive constant A_3 such that the solution $x(t)$ of Eq. (1) and its derivative satisfy

$$|x(t)| \leq A_3,\ |x'(t)| \leq A_3$$

for all $t \in I$.

Proof. To prove Theorem 4, we use the Lyapunov function $V(t,x,y)$ which is given in Eq. (4). For the case $p(t,x,y) \neq 0$, and applying the assumption of Theorem 2, we can revise the result of Lemma 1 as follows:

$$\begin{aligned} \frac{d}{dt}V(t,x,y) &\leq -A_2(x^2+y^2)+\frac{y}{h(y)}|p(t,x,y)| \\ &\leq |y|e(t). \end{aligned}$$

Using the fast that $|y| < 1+y^2$, then

$$\begin{aligned} \frac{d}{dt}V(t,x,y) &\leq (1+y^2)e(t) \\ &\leq [1+A_1^{-1}V(t,x,y)]e(t). \end{aligned}$$

Therefore

$$V'(t,x,y) \leq e(t)+A_1^{-1}V(t,x,y)e(t), \tag{5}$$

where $y^2 \leq A_1^{-1}V(t,x,y)$. Integrating inequality (5) from 0 to t and using the Gronwall-Bellman lemma, we have

$$\begin{aligned} V(t,x,y) &\leq V(0,x(0),y(0))+\int_0^t e(\tau)d\tau+A_1^{-1}\int_0^t V(\tau,x(\tau),y(\tau))e(\tau)d\tau \\ &\leq V(0,x(0),y(0))+\int_0^\infty e(\tau)d\tau+A_1^{-1}\int_0^\infty V(\tau,x(\tau),y(\tau))e(\tau)d\tau \\ &\leq (V(0,x(0),y(0))+B)\exp(A_1^{-1}B), \end{aligned} \tag{6}$$

where $B = \int_0^\infty e(\tau)d\tau$.

In view of the last inequality which satisfies the boundedness of solutions of Eq. (1), then the proof of Theorem 4 is now complete.

4. EXAMPLES

Example 1. Consider the second order differential equation (1) in which the $p(t,x,x') \equiv 0$, we have

$$x''+(1+t)x'e^{x^2}+(1+e^{-t^2})xe^{(x')^2}=0. \tag{7}$$

The equivalent system of (7) can be written as follows:

$$\begin{aligned} x' &= y, \\ y' &= -(1+t)ye^{x^2} - (1+e^{-t^2})xe^{y^2}. \end{aligned} \tag{8}$$

It is obvious that Eq. (8) satisfies the conditions (c_1) and (c_2) and using the scalar function

$$V(t,x,y) = (1+e^{-t^2}) \int_0^x \tau d\tau + \int_0^y \tau e^{-\tau^2} d\tau,$$

where $c(t) = 1 + e^{-t^2}$, $\int_0^x g(\tau)d\tau = \int_0^x \tau d\tau$ and

$\int_0^y \frac{\tau}{h(\tau)} d\tau = \int_0^y \tau e^{-\tau^2} d\tau.$

The differentiation of $V \equiv V(t,x,y)$ along the solution path of Eq. (8) gives

$$\begin{aligned} V' &= -2te^{-t^2} \int_0^x \tau d\tau - (1+t)y^2 e^{x^2-y^2} \\ &= -te^{-t^2} x^2 - (1+t)y^2. \end{aligned}$$

Thus, there exist a positive constant δ small enough such that

$$V' \leq -\delta(x^2+y^2)$$

which shows that the asymptotic stability is now established.

Example 2. In addition to Example 1 and following from Eq. (1), we examine the case for which $p(t,x,x') = \frac{4e^{x'^2}}{1+t^4+x^2+x'^2} \neq 0$. Hence, with respect to Theorem 4 we have that

$$|p(t,x,y)| \leq \frac{4}{1+t^4} e^{y^2}$$

. The result follows from inequalities (5) to (6) above that

$$V'(t,x,y) \leq \frac{4}{1+t^4} + A_1^{-1} V(t,x,y) \frac{4}{1+t^4},$$

Then, integrating the above and applying Gronswall-Bellman lemma, we have

$$V(t,x,y) \leq (V(0,x(0),y(0)) + \pi\sqrt{2}) \exp(A_1^{-1}\pi\sqrt{2}),$$

where $\int_0^\infty \frac{4}{1+s^4} ds = \pi\sqrt{2}.$

Thus, this satisfies the boundedness of solutions.

Remark. In this work, we have been able to established the asymptotic stability and boundedness of solution $x(t)$ and its derivative by applying the Lyapunov direct method. This extend and improve on the results obtained by Athanassov [1] on Eq. (1).

5. Acknowledgements

The authors appreciate the anonymous reviewers for their comments and suggestions which have improved the final version of this paper.

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